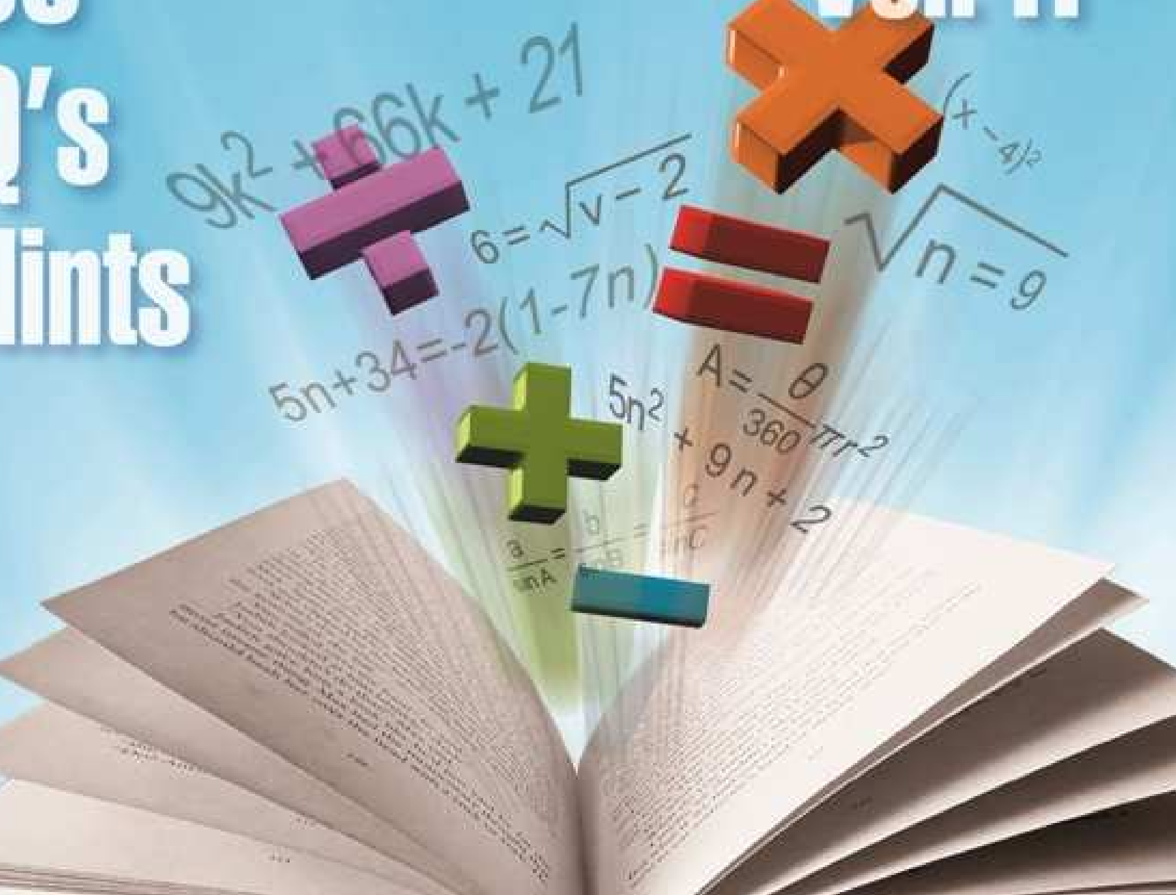
JEE - Main

# Mathematics

For all Engineering Entrance Examinations held across India.

## 2953 MCQ's with Hints

## Vol. II





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# JEE – Main
# Mathematics
## Vol. II

### Salient Features

- Exhaustive coverage of MCQs subtopic wise.
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- Hints provided wherever relevant.
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# Preface

Mathematics is the study of quantity, structure, space and change. It is one of the oldest academic discipline that has led towards human progress. Its root lies in man's fascination with numbers.

Maths not only adds great value towards a progressive society but also contributes immensely towards other sciences like Physics and Chemistry. Interdisciplinary research in the above mentioned fields has led to monumental contributions towards progress in technology.

Target's "Maths Vol. II" has been compiled according to the notified syllabus for JEE (Main), which in turn has been framed after reviewing various national syllabi.

Target's "Maths Vol. II" comprises of a comprehensive coverage of theoretical concepts and multiple choice questions. In the development of each chapter we have ensured the inclusion of shortcuts and unique points represented as an 'Important Note' for the benefit of students.

The flow of content and MCQs has been planned keeping in mind the weightage given to a topic as per the JEE (Main).

MCQs in each chapter are a mix of questions based on theory and numericals and their level of difficulty is at par with that of various engineering competitive examinations.

This edition of "Maths Vol. II" has been conceptualized with absolute focus on the assistance students would require to answer tricky questions and would give them an edge over the competition.

Lastly, I am grateful to the publishers of this book for their persistent efforts, commitment to quality and their unending support to bring out this book, without which it would have been difficult for me to partner with students on this journey towards their success.

*All the best to all Aspirants!*

Yours faithfully,
Author

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# 01 Matrices and Determinants

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- 1.1.3 Test of consistency and solution of simultaneous linear equations in two or three variables

1.2 Matrices

- 1.2.1 Matrices of order two and three, Algebra of matrices and Types of matrices
- 1.2.2 Adjoint and Evaluation of inverse of a square matrix using determinants and elementary transformations
- 1.2.3 Test of consistency and solution of simultaneous linear equations in two or three variables

## 1.1 Determinants

### 1. Determinant of order two and three:

**i. Determinant of order two:**

The arrangement of four numbers $a_1$, $a_2$, $b_1$, $b_2$ in 2 rows and 2 columns enclosed between two vertical bars as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a **determinant of order two**.

The value of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is defined as $a_1b_2 - b_1a_2$.

**eg.**

$$\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix} = 2(7) - (-3)4$$
$$= 14 + 12$$
$$= 26$$

**ii. Determinant of order three:**

The arrangement of nine numbers $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$, $a_3$, $b_3$, $c_3$ in 3 rows and 3 columns enclosed between two vertical bars as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a **determinant of order three**.

Here, the elements in the horizontal line are said to form a **row**. The three rows are denoted by $R_1$, $R_2$, $R_3$ respectively.

Similarly, the elements in the vertical line are said to form a **column**. The three columns are denoted by $C_1$, $C_2$, $C_3$ respectively.

The value of the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is given by

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - b_2a_3)$$

**eg.**

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 3 \\ 1 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$
$$= 1(2 - 9) - 1(6 - 3) + 2(9 - 1)$$
$$= -7 - 3 + 16$$
$$= 6$$

**Important Notes**

- A determinant of order 3 can be expanded along any row or column.
- If each element of a row or a column of a determinant is zero, then its value is zero.

### 2. Minors and Cofactors:

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Here, $a_{ij}$ denotes the element of the determinant $\Delta$ in $i^{th}$ row and $j^{th}$ column.

**i. Minor of an element:**

The **minor** of an element $a_{ij}$ is defined as the value of the determinant obtained by eliminating the $i^{th}$ row and $j^{th}$ column of $\Delta$.

It is denoted by $M_{ij}$.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$M_{11}$ = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

$M_{12}$ = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

$M_{13}$ = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Similarly, we can find the minors of other elements.

**eg.**

Find the minor of 2 in the determinant $\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 1 & 6 & 7 \end{vmatrix}$.

**Solution:** Minor of 2 $= \begin{vmatrix} 0 & 5 \\ 6 & 7 \end{vmatrix} = (0)(7) - (5)(6) = -30$

**ii. Cofactor of an element:**

The **cofactor** of an element $a_{ij}$ in $\Delta$ is equal to $(-1)^{i+j} M_{ij}$, where $M_{ij}$ is the minor of $a_{ij}$.

It is denoted by $C_{ij}$ or $A_{ij}$.

Thus, $C_{ij} = (-1)^{i+j} M_{ij}$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$C_{11} = (-1)^{1+1} M_{11} = M_{11}$

$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$C_{13} = (-1)^{1+3} M_{13} = M_{13}$

Similarly, we can find the cofactors of other elements.

**eg.**

Find the cofactor of 3 in the determinant $\begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ 1 & 6 & 7 \end{vmatrix}$.

**Solution:** Cofactor of 3 $= (-1)^{1+2} \begin{vmatrix} 4 & 5 \\ 1 & 7 \end{vmatrix}$

$= -(28 - 5)$

$= -23$

**Important Note**

❖ The sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of any other row (or column) is zero.

**3. Properties of determinants:**

i. The value of a determinant is unchanged, if its rows and columns are interchanged.

Thus, if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then the value of $D_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is D.

**eg.**

Let $D = \begin{vmatrix} 8 & -5 & 1 \\ 5 & 8 & 1 \\ 6 & 3 & 1 \end{vmatrix}$

By interchanging rows and columns, we get

$D_1 = \begin{vmatrix} 8 & 5 & 6 \\ -5 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

By expanding the determinants D and $D_1$, we get the value of each determinant equal to 2.

ii. Interchanging of any two rows (or columns) will change the sign of the value of the determinant.

**eg.**

Let $D = \begin{vmatrix} 8 & -5 & 1 \\ 5 & 8 & 1 \\ 6 & 3 & 1 \end{vmatrix}$. Then, D = 2

Let $D_1$ be the determinant obtained by interchanging second and third row of D. Then,

$D_1 = \begin{vmatrix} 8 & -5 & 1 \\ 6 & 3 & 1 \\ 5 & 8 & 1 \end{vmatrix}$

$= -2$

$\therefore$ $D_1 = -D$

iii. If any two rows (or columns) of a determinant are identical, then its value is zero.

**eg.**

a. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$ ....[$\because R_1 \equiv R_3$]

b. $\begin{vmatrix} 41 & 1 & 1 \\ 44 & 1 & 1 \\ 47 & 1 & 1 \end{vmatrix} = 0$ ....[$\because C_2 \equiv C_3$]

iv. If all the elements of any row (or column) are multiplied by a number k, then the value of new determinant so obtained is k times the value of the original determinant.

**eg.**

Let $D = \begin{vmatrix} 8 & 5 & 6 \\ -5 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix}$. Then, D = 2

Let $D_1$ be the determinant obtained by multiplying the third row of D by k. Then,

$D_1 = \begin{vmatrix} 8 & 5 & 6 \\ -5 & 8 & 3 \\ k & k & k \end{vmatrix} = 8(5k) - 5(-8k) + 6(-13k)$

$= 2k = kD$

v. If each element of any row (or column) of a determinant is the sum of two terms, then the determinant can be expressed as the sum of two determinants.

**eg.**

a. $$\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

b. $$\begin{vmatrix} a_1 & b_1+p & c_1 \\ a_2 & b_2+q & c_2 \\ a_3 & b_3+r & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & p & c_1 \\ a_2 & q & c_2 \\ a_3 & r & c_3 \end{vmatrix}$$

vi. If a constant multiple of all elements of any row (or column) is added to the corresponding elements of any other row (or column), then the value of the new determinant so obtained remains unchanged.

**eg.**

Let $D = \begin{vmatrix} 8 & 5 & 6 \\ -5 & 8 & 3 \\ 1 & 1 & 1 \end{vmatrix}$.

Let $D_1$ be the determinant obtained by multiplying the elements of the first row of D by k and adding these elements to the corresponding elements of the third row. Then,

$$D_1 = \begin{vmatrix} 8 & 5 & 6 \\ -5 & 8 & 3 \\ 1+8k & 1+5k & 1+6k \end{vmatrix}$$

By solving, we get $D = D_1$

**4. Product or Multiplication of two determinants:**

Let the two determinants of third order be $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$ and $\Delta$ be their product.

**Rule:** Take the 1st row of $\Delta_1$ and multiply it successively with 1st, 2nd and 3rd rows of $\Delta_2$. The three expressions thus obtained will be elements of 1st row of $\Delta$. In a similar manner the elements of 2nd and 3rd row of $\Delta$ are obtained.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

This is row by row multiplication rule for finding the product of two determinants.
We can also multiply rows by columns or columns by rows or columns by columns.

**eg.**

Find the value of $\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}$.

**Solution:** $\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 17 & 9 \\ 9 & 5 \end{vmatrix} = 4$

5. **Area of a triangle using determinants:**

Area of a triangle whose vertices are $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ is equal to $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Since, area cannot be negative, therefore we always take the absolute value of the above determinant for the area.

**eg.**

If the vertices of a triangle are (3, 3), (–5, 7) and (–1, 4), then find its area.

**Solution:** Let A ≡ (3, 3), B ≡ (–5, 7) and C ≡ (–1, 4)

$\therefore$ area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 3 & 3 & 1 \\ -5 & 7 & 1 \\ -1 & 4 & 1 \end{vmatrix}$

= 4 sq.units

**Important Notes**

- If points $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.
- The equation of the line joining points $(x_1, y_1)$ and $(x_2, y_2)$ is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

6. **System of linear equations:**

A system of linear equations in 3 unknowns $x$, $y$, z is of the form

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

i. If $d_1$, $d_2$ and $d_3$ are all zero, the system is called **homogeneous** and **non–homogeneous** if at least one $d_i$ is non-zero.

ii. A system of linear equations may have a unique solution, or many solutions, or no solution at all. If it has a solution (whether unique or not) the system is said to be **consistent**. If it has no solution, it is called an **inconsistent** system.

7. **Solution of a non-homogeneous system of linear equations:**

i. The solution of the system of linear equations

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

provided that $D \neq 0$.

**Conditions for consistency:**

a. If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$.

b. If D = 0 and $D_1 = D_2 = 0$, then the given system of equations is consistent and has infinitely many solutions.

c. If D = 0 and one of $D_1$ and $D_2$ is non-zero, then the given system of equations is inconsistent.

ii. The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$,

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

provided that $D \neq 0$.

**Conditions for consistency:**

a. If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

b. If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is either consistent with infinitely many solutions or has no solution.

c. If $D = 0$ and at least one of the determinants $D_1$, $D_2$ and $D_3$ is non-zero, then the given system of equations is inconsistent.

**egs.**

i. Using Cramer's Rule, solve the following linear equations:

$3x - 2y = 5,$

$x - 3y = -3$

**Solution:** We have, $D = \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} = -7 \neq 0$

$D_1 = \begin{vmatrix} 5 & -2 \\ -3 & -3 \end{vmatrix} = -21$ and $D_2 = \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} = -14$

∴ by cramer's rule,

$x = \frac{D_1}{D} = \frac{-21}{-7} = 3$

and $y = \frac{D_2}{D} = \frac{-14}{-7} = 2$

∴ $x = 3$ and $y = 2$.

ii. Verify whether the system of equations: $3x - y - 2z = 2$, $2y - z = -1$, $3x - 5y = 3$ is consistent or inconsistent.

**Solution:** We have, $D = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = 0$, $D_1 = \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} = -5 \neq 0$

Since, $D = 0$ and $D_1 \neq 0$

Hence, the given system of equations is inconsistent.

**8. Solution of a homogeneous system of linear equations:**

If $a_1x + b_1y + c_1z = 0$
$a_2x + b_2y + c_2z = 0$
$a_3x + b_3y + c_3z = 0$
is a homogeneous system of equations, such that

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then $x = y = z = 0$ is the only solution and it is known as the trivial solution.

If D = 0, then the system is consistent with infinitely many solutions.

**eg.**

Solve the following system of homogeneous equations:

$3x - 4y + 5z = 0$
$x + y - 2z = 0$
$2x + 3y + z = 0$

**Solution:** $D = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 46 \neq 0$

$\therefore$ the given system of equations has only the trivial solution i.e., $x = y = z = 0$.

## 1.2 Matrices

**1. Matrix:**

A rectangular arrangement of mn numbers (real or complex) in m rows and n columns is called a **matrix**. This arrangement is enclosed by [ ] or ( ). Generally matrices are represented by capital letters A, B, C, etc. and its elements are represented by small letters a, b, c, etc.

**2. Order of a matrix:**

If a matrix A has m rows and n columns, then A is of **order** m × n or simple m × n matrix (read as m by n matrix). A matrix A of order m × n is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

or $A = [a_{ij}]_{m \times n}$, where i = 1, 2, …. m
j = 1, 2, …. n

Here, $a_{ij}$ denotes the element of the matrix A in $i^{th}$ row and $j^{th}$ column.

A matrix of order m × n contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

**eg.**

Order of the matrix $\begin{bmatrix} 3 & -1 \\ 2 & 3 \\ 4 & -7 \end{bmatrix}$ is 3 × 2.

**3. Types of matrices:**

**i. Row matrix:**

A matrix having only one row is called a **row-matrix** or a **row vector**.

Thus, $A = [a_{ij}]_{m \times n}$ is a row matrix, if m = 1.

**eg.**

$[3], [5 \quad 2 \quad 3]$ are row matrices of order 1 × 1, 1 × 3 respectively.

**ii. Column matrix:**

A matrix having only one column is called a **column matrix** or a **column vector**.

Thus, $A = [a_{ij}]_{m \times n}$ is a column matrix, if n = 1

**eg.**

$[3], \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ are column matrices of order 1 × 1, 3 × 1 respectively.

**iii. Rectangular matrix:**

A matrix $A = [a_{ij}]_{m\times n}$ is called a **rectangular matrix**, if number of rows is not equal to number of columns (m ≠ n).

**eg.**

$[3 \quad -2 \quad 1], \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 4 & -3 \end{bmatrix}$ are rectangular matrices of order 1 × 3, 3 × 2 respectively.

**iv. Square matrix:**

A matrix $A = [a_{ij}]_{m\times n}$ is called a **square matrix**, if number of rows is equal to number of columns (m = n).

**eg.**

$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ are square matrices of order 2 × 2, 3 × 3 respectively.

**v. Null matrix or Zero matrix:**

A matrix whose all elements are zero is called a **null matrix** or a **zero matrix**. It is denoted by O.

Thus, $A = [a_{ij}]_{m \times n}$ is a zero matrix, if $a_{ij} = 0 \ \forall$ i and j

**eg.**

$[0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are zero matrices of order 1 × 1, 2 × 2 respectively.

**vi. Diagonal matrix:**

A square matrix in which all its non-diagonal elements are zero is called a **diagonal matrix**.

Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is a diagonal matrix, if $a_{ij} = 0 \ \forall \ i \neq j$

**eg.**

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are diagonal matrices of order 2 and 3 respectively.

**Important Notes**

- A diagonal matrix of order n × n having $d_1, d_2, \ldots, d_n$ as diagonal elements is denoted by diag $[d_1, d_2, \ldots, d_n]$.

  **eg.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix and is denoted by diag [1, 4, 7].

- Number of zeros in a diagonal matrix of order n is $n^2 - n$.

**vii. Scalar matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called a **scalar matrix**, if all its non-diagonal elements are zero and diagonal elements are same.

Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is a scalar matrix, if $a_{ij} = \begin{cases} 0, & i \neq j \\ \lambda, & i = j \end{cases}$, where $\lambda$ is a constant.

**eg.**

$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are scalar matrices of order 2 and 3 respectively.

**Important Note**

❖ A scalar matrix is always a diagonal matrix.

**viii. Unit matrix or Identity matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called an **identity** or **unit matrix**, if all its non-diagonal elements are zero and diagonal elements are one.

Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is a unit matrix, if $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

A unit matrix is denoted by I.

**eg.**

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 2 and 3 respectively.

**Important Note**

❖ Every unit matrix is a diagonal as well as a scalar matrix.

**ix. Triangular matrix:**

A square matrix is said to be **triangular matrix** if each element above or below the diagonal is zero.

**a. Upper triangular matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called an **upper triangular matrix**, if every element below the diagonal is zero.

Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is an upper triangular matrix, if $a_{ij} = 0 \ \forall \ i > j$

**eg.**

$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 9 \end{bmatrix}$ is an upper triangular matrix.

**b. Lower triangular matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called a **lower triangular matrix**, if every element above the diagonal is zero.

Thus, a square matrix $A = [a_{ij}]_{n\times n}$ is a lower triangular matrix, if $a_{ij} = 0 \ \forall \ i < j$

**eg.**

$\begin{bmatrix} 5 & 0 & 0 \\ 2 & 6 & 0 \\ 1 & 3 & 4 \end{bmatrix}$ is a lower triangular matrix.

**Important Notes**

- ❖ Minimum number of zeros in a triangular matrix of order n is $\frac{n(n-1)}{2}$.
- ❖ A diagonal matrix is an upper as well as lower triangular matrix.

**x. Singular matrix:**

A square matrix A is called a singular matrix, if |A| = 0.

**eg.**

If $A = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$, then

$|A| = \begin{vmatrix} -3 & 3 \\ 1 & -1 \end{vmatrix}$

$= 3 - 3 = 0$

∴ A is a singular matrix.

**xi. Non-singular matrix:**

A square matrix A is called a non-singular matrix, if $|A| \neq 0$.

**eg.**

If $A = \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix}$, then

$|A| = \begin{vmatrix} 5 & -3 \\ 2 & 4 \end{vmatrix} = 20 + 6 = 26 \neq 0$

∴ A is a non-singular matrix.

**4. Trace of a matrix:**

The sum of all diagonal elements of a square matrix A is called the **trace** of matrix A. It is denoted by tr(A).

Thus, $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$

**eg.**

If $A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix}$, then tr(A) = 3 + 4 + 8 = 15

**Properties of trace of a matrix:**

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ and λ be a scalar. Then,

i. $tr(A \pm B) = tr(A) \pm tr(B)$

ii. $tr(\lambda A) = \lambda\, tr(A)$

iii. $tr(AB) = tr(BA)$

iv. $tr(A) = tr(A^T)$

v. $tr(I_n) = n$

vi. $tr(O) = 0$

vii. $tr(AB) \neq tr(A).\, tr(B)$

**5. Submatrix:**

A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a **submatrix** of the given matrix.

**eg.**

$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ is a submatrix of the matrix $\begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & -1 \\ 6 & 3 & 5 \end{bmatrix}$.

**6. Equality of matrices:**

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if

i. they are of the same order

ii. their corresponding elements are equal (i.e., $a_{ij} = b_{ij}\ \forall\ i, j$)

**egs.**

a. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ are equal matrices, then
$a = 1, b = 2, c = 3$ and $d = 4$

b. $C = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ are not equal matrices because their orders are not same.

**7. Algebra of matrices:**

**i. Addition of matrices:**

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices. Then their sum (denoted by A + B) is defined to be matrix $[c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

**eg.**

If $A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -8 & 9 \\ 2 & 8 & -4 \end{bmatrix}$,

then $A + B = \begin{bmatrix} 1+7 & 2-8 & -3+9 \\ 4+2 & -5+8 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 6 \\ 6 & 3 & 2 \end{bmatrix}$

Similarly, their **subtraction** A – B is defined as $A - B = [a_{ij} - b_{ij}]_{m \times n}\ \forall\ i, j$.

**Important Note**

❖ Matrix addition and subtraction can be possible only when matrices are of the same order.

**Properties of matrix addition:**

If A, B and C are three matrices of same order, then

a. $A + B = B + A$ (Commutative law)

b. $(A + B) + C = A + (B + C)$ (Associative law)

c. $A + O = O + A = A$, where O is a zero matrix of the same order as A.

d. $A + (-A) = O = (-A) + A$, where (–A) is obtained by changing the sign of every element of A which is additive inverse of the matrix.

e. $\left.\begin{matrix} A + B = A + C \\ B + A = C + A \end{matrix}\right\} \Rightarrow B = C$ (Cancellation law)

**ii. Multiplication of matrices:**

Let A and B be any two matrices, then their product AB will be defined only when number of columns in A is equal to the number of rows in B. If $A = [a_{ik}]_{m\times n}$ and $B = [b_{kj}]_{n\times p}$, then their product AB is of order m × p and is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$
$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$
$$= (i^{th} \text{ row of A}) (j^{th} \text{ column of B})$$

**eg.**

Find AB, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$.

**Solution:** Here, number of columns of A = 3 = number of rows of B.

∴ AB is defined as a 2 × 3 matrix.

$$\therefore \quad AB = \begin{bmatrix} 1\times2+2\times3+3\times4 & 1\times5+2\times6+3\times7 & 1\times3+2\times4+3\times5 \\ 4\times2+5\times3+6\times4 & 4\times5+5\times6+6\times7 & 4\times3+5\times4+6\times5 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}$$

**Properties of matrix multiplication:**

If A, B and C are three matrices such that their product is defined, then

a. AB ≠ BA (generally not commutative)

b. (AB)C = A(BC) (Associative law)

c. AI = IA = A, where A is a square matrix and I is an identity matrix of same order.

d. A (B + C) = AB + AC
(A + B)C = AC + BC } (Distributive law)

e. AB = AC ⇏ B = C (Cancellation law is not applicable)

f. If AB = 0, then it does not imply that A = 0 or B = 0

**Important Notes**

- Multiplication of two diagonal matrices is a diagonal matrix.
- Multiplication of two scalar matrices is a scalar matrix.
- If A and B are square matrices of the same order, then
  - i. $(A + B)^2 = A^2 + AB + BA + B^2$
  - ii. $(A - B)^2 = A^2 - AB - BA + B^2$
  - iii. $(A+B)(A-B) = A^2 - AB + BA - B^2$
  - iv. $A(-B) = (-A)B = -(AB)$
- $(A + B)^2 \neq A^2 + 2AB + B^2$ unless AB = BA

**a. Scalar multiplication of matrices:**

Let $A = [a_{ij}]_{m\times n}$ be a matrix and λ be a number (scalar), then the matrix obtained by multiplying every element of A by λ is called the **scalar multiple** of A by λ. It is denoted by λA.

Thus, if $A = [a_{ij}]_{m \times n}$, then $\lambda A = A\lambda = [\lambda a_{ij}]_{m \times n}$ ∀i, j

**Properties of scalar multiplication:**

If A, B are two matrices of same order and $\alpha$, $\beta$ are any numbers, then

1. $\alpha(A \pm B) = \alpha A \pm \alpha B$
2. $(\alpha \pm \beta)A = \alpha A \pm \beta A$
3. $\alpha(\beta A) = \beta(\alpha A) = (\alpha\beta A)$
4. $(-\alpha)A = -(\alpha A) = \alpha(-A)$
5. $O.A = O$
6. $\alpha.O = O$

**b. Positive integral powers of matrices:**

The positive integral powers of a matrix A are defined only when A is a square matrix.

Then, we define

$A^1 = A$ and $A^{n+1} = A^n.A$, where $n \in N$

From this definition,

$A^2 = A.A$, $A^3 = A^2.\ A = A.A.A$

For any positive integers m and n,

1. $A^m\ A^n = A^{m+n}$
2. $(A^m)^n = A^{mn} = (A^n)^m$
3. $I^n = I$, $I^m = I$
4. $A^0 = I_n$, where A is a square matrix of order n

**8. Transpose of a matrix:**

A matrix obtained from the matrix A by interchanging its rows and columns is called the **transpose** of A. It is denoted by $A^t$ or $A^T$ or $A'$.

Thus, if the order of A is $m \times n$, then the order of $A^T$ is $n \times m$.

**eg.**

If $A = \begin{bmatrix} 2 & 4 & -1 \\ 3 & -1 & 2 \end{bmatrix}$, then $A^T = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ -1 & 2 \end{bmatrix}$

**Properties of transpose of a matrix:**

If A and B are two matrices, then

i. $(A^T)^T = A$

ii. $(kA)^T = kA^T$, where k is a scalar

iii. a. $(A + B)^T = A^T + B^T$
b. $(A - B)^T = A^T - B^T$
} A and B being of the same order

iv. $(AB)^T = B^T A^T$, A and B being conformable for the product AB

v. $(A^n)^T = (A^T)^n$, $n \in N$

vi. a. Trace $A^T$ = Trace A

b. Trace $AA^T \geq 0$

**9. Symmetric matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called **symmetric matrix**, if $A = A^T$ or $a_{ij} = a_{ji}$ $\forall$ i, j.

**eg.**

If $A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$, then

$A^T = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$

$\therefore$ $A = A^T$

$\therefore$ A is a symmetric matrix.

**Important Notes**

- ❖ A unit matrix is always a symmetric matrix.
- ❖ Maximum number of different elements in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

**10. Skew-symmetric matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called **skew-symmetric matrix**, if $A = -A^T$ or $a_{ij} = -a_{ji}$ $\forall$ $i_j$ j.

**eg.**

If $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, then

$A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$\therefore$ $A^T = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$\therefore$ $A = -A^T$

$\therefore$ A is a skew-symmetric matrix.

**Important Notes**

- ❖ All diagonal elements of a skew-symmetric matrix are always zero.
- ❖ Trace of a skew-symmetric matrix is always zero.

**Properties of symmetric and skew-symmetric matrices:**

i. If A is a square matrix, then $A + A^T$, $AA^T, A^TA$ are symmetric matrices and $A - A^T$ is a skew-symmetric matrix.

ii. The matrix $B^TAB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

iii. If A is a skew-symmetric matrix, then
   a. $A^{2n}$ is a symmetric matrix for $n \in N$.
   b. $A^{2n+1}$ is a skew-symmetric matrix for $n \in N$.

iv. If A and B are symmetric matrices, then
   a. $A \pm B$, $AB + BA$ are symmetric matrices
   b. $AB - BA$ is a skew- symmetric matrix
   c. AB is a symmetric matrix iff $AB = BA$.

v. If A and B are skew-symmetric matrices, then
   a. $A \pm B$, $AB - BA$ are skew-symmetric matrices.
   b. $AB + BA$ is a symmetric matrix.

vi. A square matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrix as

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$$

**11. Orthogonal matrix:**

A square matrix A is called an **orthogonal** matrix, if $AA^T = A^TA = I$

**eg.**

If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore$ $A.A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Similarly, $A^TA = I$

$\therefore$ A is an orthogonal matrix.

**Important Note**

❖ A unit matrix is always a orthogonal matrix.

**Properties of orthogonal matrix:**

i. If A is an orthogonal matrix, then $A^T$ and $A^{-1}$ are also orthogonal matrices.

ii. If A and B are two orthogonal matrices, then AB and BA are also orthogonal matrices.

**12. Idempotent matrix:**

A square matrix A is called an **idempotent** matrix, if $A^2 = A$.

**eg.**

If $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then

$A^2 = A.A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$

$\therefore$ A is an idempotent matrix.

**Important Note**

❖ A unit matrix is always an idempotent matrix.

**13. Nilpotent matrix:**

A square matrix A is said to be a **nilpotent** matrix of index p, if p is the least positive integer such that $A^p = O$.

**eg.**

If $A = \begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix}$, then

$A^2 = A.A = \begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore$ A is a nilpotent matrix of index 2.

**Important Note**

❖ Determinant of every nilpotent matrix is zero.

**14. Involutory matrix:**

A square matrix A is said to be an **involutory matrix**, if $A^2 = I$.

**eg.**

If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

∴ A is an involutory matrix.

**Important Note**

❖ Every unit matrix is involutory.

**15. Conjugate of a matrix:**

The matrix obtained from a given matrix A by replacing each entry containing complex numbers with its complex conjugate is called **conjugate** of A. It is denoted by $\overline{A}$.

**eg.**

If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, then $\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

**Transpose conjugate of a matrix:**

The transpose of the conjugate of a matrix A is called **transpose conjugate** of A.

It is denoted by $A^\theta$.

**eg.**

If $A = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$, then $A^\theta = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}$

**16. Hermitian matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be **hermitian matrix**, if $A = A^\theta$ or $a_{ij} = \overline{a}_{ji}$ $\forall$ i, j.

**eg.**

If $A = \begin{bmatrix} 3 & 3+4i \\ 3-4i & 5 \end{bmatrix}$, then $A^\theta = \begin{bmatrix} 3 & 3+4i \\ 3-4i & 5 \end{bmatrix}$

∴ $A = A^\theta$

∴ A is a hermitian matrix.

**Important Note**

❖ Determinant of a hermitian matrix is purely real.

**17. Skew-Hermitian matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be a **skew-hermitian** matrix if $A = -A^\theta$ or $a_{ij} = -\bar{a}_{ji}\ \ \forall$ i, j

**eg.**

If $A = \begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$,

then $A^\theta = \begin{bmatrix} -i & -1+2i \\ 1+2i & 0 \end{bmatrix}$

$= -\begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$

∴ $A = -A^\theta$

∴ A is a skew-hermitian matrix.

**18. Adjoint of a square matrix:**

The **adjoint** of a square matrix $A = [a_{ij}]$ is the transpose of the matrix of cofactors of elements of A. It is denoted by adj A.

Let $A = [a_{ij}]$ be a square matrix and $C_{ij}$ be the cofactor of $a_{ij}$ in A. Then, adj $A = [C_{ij}]^T$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

**eg.**

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, find adj A.

**Solution:** Here, $C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = 6$, $C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = -2$,

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$, $C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1$,

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -5$, $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3$,

$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -5$, $C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = 4$,

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

∴ adj $A = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

**Properties of adjoint matrix:**

If A and B are square matrices of order n such that $|A| \neq 0$ and $|B| \neq 0$, then

i. $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A$

ii. $|\text{adj } A| = |A|^{n-1}$

iii. $\text{adj }(\text{adj } A) = |A|^{n-2} A$

iv. $|\text{adj (adj A)}| = |A|^{(n-1)^2}$

v. $\text{adj}(A^T) = (\text{adj } A)^T$

vi. adj (AB) = (adj B) (adj A)

vii. $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in N$

viii. $\text{adj}(kA) = k^{n-1}(\text{adj } A)$, $k \in R$

ix. $\text{adj}(I_n) = I_n$

x. adj (O) = O

**Important Notes**

- Adjoint of a diagonal matrix is a diagonal matrix.
- Adjoint of a triangular matrix is a triangular matrix.
- Adjoint of a singular matrix is a singular matrix.
- Adjoint of a symmetric matrix is a symmetric matrix.

**19. Inverse of a matrix:**

Let A be a n-rowed square matrix. Then, if there exists a square matrix B of the same order such that AB = I = BA, matrix B is called the **inverse** of matrix A.

It is denoted by $A^{-1}$.

Thus, $AA^{-1} = I = A^{-1}A$

A square matrix A has inverse iff A is non-singular i.e., $A^{-1}$ exists iff $|A| \neq 0$.

**Inverse by adjoint method:**

The **inverse** of a non-singular square matrix A is given by $A^{-1} = \frac{1}{|A|}(\text{adj} A)$, if $|A| \neq 0$.

**eg.**

If $A = \begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix}$, find $A^{-1}$.

**Solution:** $|A| = \begin{vmatrix} 2 & -3 \\ -4 & 2 \end{vmatrix} = -8 \neq 0$

$\therefore$ $A^{-1}$ exists

Here, $C_{11} = (-1)^{1+1}(2) = 2$, $C_{12} = (-1)^{1+2}(-4) = 4$,

$C_{21} = (-1)^{2+1}(-3) = 3$, $C_{22} = (-1)^{2+2}(2) = 2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj} A}{|A|} = -\frac{1}{8}\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

**Important Notes**

- Matrix A is invertible if $A^{-1}$ exists.
- The inverse of a square matrix, if exists, it is unique.
- A nilpotent matrix is always non - invertible.

**Properties of inverse matrix:**

If A and B are invertible matrices of the same order, then

i. $(A^{-1})^{-1} = A$

ii. $(A^T)^{-1} = (A^{-1})^T$

iii. $(AB)^{-1} = B^{-1} A^{-1}$

iv. $(A^n)^{-1} = (A^{-1})^n$, $n \in N$

v. $\text{adj}\,(A^{-1}) = (\text{adj}A)^{-1}$

vi. $|A^{-1}| = \dfrac{1}{|A|}$

**Important Notes**

- Inverse of a diagonal matrix is a diagonal matrix.
- Inverse of a triangular matrix is a triangular matrix.
- Inverse of a scalar matrix is a scalar matrix.
- Inverse of a symmetric matrix is a symmetric matrix.

**20. Elementary transformations:**

The **elementary transformations** are the operations performed on rows (or columns) of a matrix.

i. Interchanging any two rows (or columns). It is denoted by $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$).

ii. Multiplying the elements of any row (or column) by a non-zero scalar.
It is denoted by $R_i \to kR_i$ ($C_i \to kC_i$)

iii. Multiplying the elements of any row (or column) by a non-zero scalar k and adding them to corresponding elements of another row (or column).
It is denoted by $R_i + kR_j$($C_i + kC_j$).

**Inverse of a non-singular square matrix by elementary transformations:**

Let A be a non-singular square matrix of order n.

**To find $A^{-1}$ by elementary row (or column) transformations:**

i. Consider, $AA^{-1} = I$

ii. Perform suitable elementary row (or column) transformations on matrix A, so as to convert it into an identity matrix of order n.

iii. The same row (or column) transformations should be performed on the R.H.S. i.e. on I. Let, I gets converted into a $n \times n$ matrix B.

iv. Thus, $AA^{-1} = I$ reduces to $IA^{-1} = B$ i.e. $A^{-1} = B$.

**eg.**

If $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, find $A^{-1}$.

**Solution:** $|A| = \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0$

$\therefore$ $A^{-1}$ exists.

Consider, $AA^{-1} = I$

$\therefore$ $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, we get

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

**21. Rank of a matrix:**

A positive integer r is said to be the **rank** of a non – zero matrix A, if

i. there exists at least one minor in A of order r which is not zero and

ii. every minor of order (r + 1) or more is zero.
It is denoted by $\rho(A) = r$.

**eg.**

Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

**Solution:** $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} = 1(1) - 2(2) + 3(1) = 0$

$\therefore$ rank of $A < 3$

$\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1 \neq 0$

$\therefore$ rank of A = 2.

**Important Note**

❖ The rank of the null matrix is not defined and the rank of every non-null matrix is greater than or equal to 1.

**Properties of rank of a matrix:**

i. If $I_n$ is a unit matrix of order n, then $\rho(I_n) = n$.

ii. If A is a $n \times n$ non-singular matrix, then $\rho(A) = n$.

iii. The rank of a singular square matrix of order n cannot be n.

iv. Elementary operations do not change the rank of a matrix.

v. If $A^T$ is a transpose of A, then $\rho(A^T) = \rho(A)$.

vi. If A is an $m \times n$ matrix, then $r(A) \leq \min(m, n)$.

**22. Echelon form of a matrix:**

A non-zero matrix A is said to be in **echelon** form if either A is the null matrix or A satisfies the following conditions:

i. Every non-zero row in A precedes every zero row.

ii. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

The number of non-zero rows of a matrix given in the echelon form is its rank.

**eg.**

The matrix $\begin{bmatrix} 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in the echelon form because

it has two non-zero rows, so the rank is 2.

**23. System of simultaneous linear equations:**

Consider the following system of m linear equations in n unknowns as given below:

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

…… …… ….. …… .….

…… …… ….. …… .….

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

This system of equations can be written in matrix form as AX = B,

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

The m × n matrix A is called the **coefficient matrix** of the system of linear equations.

**i. Solution of non-homogeneous system of linear equations:**

**a. Matrix method:**

If AX = B, then $X = A^{-1}B$ gives a unique solution, provided A is non-singular. But if A is a singular matrix i.e., if |A| = 0, then the system of equation AX = B may be consistent with infinitely many solutions or it may be inconsistent.

**b. Rank method:**

Rank method for solution of non-homogeneous system AX = B

1. Write down A, B
2. Write the augmented matrix [A : B]
3. Reduce the augmented matrix to echelon form by using elementary row operations.
4. Find the number of non-zero rows in A and [A : B] to find the ranks of A and [A : B] respectively.
5. If $\rho(A) \neq \rho(A : B)$, then the system is inconsistent.
6. If $\rho(A) = \rho(A : B)$ = number of unknowns, then the system has a unique solution.

   If $\rho(A) = \rho(A : B)$ < number of unknowns, then the system has an infinite number of solutions.

**c. Criterion of consistency:**

Let AX = B be a system of n-linear equations in n unknowns.

1. If $|A| \neq 0$, then the system is consistent and has the unique solution given by $X = A^{-1}B$.
2. If $|A| = 0$ and (adj A) B = O, then the given system of equations is consistent and has infinitely many solutions.
3. If $|A| = 0$ and (adj A) B $\neq$ O, then the given system of equations is inconsistent.

**egs.**

a. Solve the following system of equations by matrix method:
$3x - 4y = 5$ and $4x + 2y = 3$

**Solution:** The given system of equations can be written in the matrix form as AX = B,

where $A = \begin{bmatrix} 3 & -4 \\ 4 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Now, $|A| = 22 \neq 0$

The given system of equations has a unique solution given by $X = A^{-1}B$.

Here, $C_{11} = 2$, $C_{12} = -4$, $C_{21} = 4$, $C_{22} = 3$

$\therefore$ $\text{adj } A = \begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{\text{adj}A}{|A|} = \dfrac{1}{22}\begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}$

$\therefore$ $X = A^{-1}B = \dfrac{1}{22}\begin{bmatrix} 2 & 4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \dfrac{1}{22}\begin{bmatrix} 22 \\ -11 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{-1}{2} \end{bmatrix}$

$\therefore$ $x = 1$ and $y = -\dfrac{1}{2}$

b. For what value of $\lambda$, the system of equations
$x + y + z = 6$,
$x + 2y + 3z = 10$,
$x + 2y + \lambda z = 12$
is inconsistent?

**Solution:** The given system of equations can be written as $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 12 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & \lambda - 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

For $\lambda = 3$, rank of matrix A is 2 and that of the augmented matrix is 3.

So, the system is inconsistent.

c. Solve the system of equations

$2x - y = 5,$

$4x - 2y = 7.$

**Solution:** The given system of equations can be written in the matrix form as AX = B,

where $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

Now, $|A| = 0$

∴ A is singular.

Either the given system of equations has no solution or an infinite number of solutions.

Here, $C_{11} = -2$, $C_{12} = -4$, $C_{21} = 1$, $C_{22} = 2$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$$

$$\therefore \quad (\text{adj } A)\, B = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -6 \end{bmatrix} \neq 0.$$

Hence, the given system of equations is inconsistent.

**24. Solution of homogeneous system of linear equations:**

**i. Matrix method:**

Let AX = O be a homogeneous system of n-linear equations with n-unknowns. If A is a non-singular matrix, then the system of equations has a unique solution X = O i.e., $x_1 = x_2 = \ldots = x_n = 0$.

This solution is known as a **trivial solution**. A system AX = O of n homogeneous linear equations in n unknowns, has **non-trivial** solution iff the coefficient matrix A is singular.

**ii. Rank method:**

In case of a homogeneous system of linear equations, the rank of the augmented matrix is always same as that of the coefficient matrix. So, **a homogeneous system of linear equations is always consistent.**

If r(A) = n = number of variables, then AX = O has a unique solution X = 0 i.e., $x_1 = x_2 = \ldots = x_n = 0$

If r(A) = r < n ( = number of variables), then the system of equations has infinitely many solutions.

**25. Properties of determinant of a matrix:**

i. If A and B are square matrices of the same order, then $|AB| = |A|\,|B|$.

ii. If A is a square matrix of order n, then $|A| = |A^T|$.

iii. If A is a square matrix of order n, then $|kA| = k^n\,|A|$.

iv. If A and B are square matrices of same order, then $|AB| = |BA|$.

v. If A is a skew-symmetric matrix of odd order, then $|A| = 0$.

vi. $|A|^n = |A^n|$, $n \in N$

**Formulae**

## 1.1 Determinants

**1. Determinant of order two and three:**

i. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then det $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}$

ii. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{31}\,a_{23} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{31}\,a_{22}$$

**2. Minors and Cofactors:**

**i. Minor of an element:**

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$M_{11}$ = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

$M_{12}$ = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

$M_{13}$ = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$ and so on.

**ii. Cofactor of an element:**

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$C_{11} = (-1)^{1+1} M_{11} = M_{11}$
$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$
$C_{13} = (-1)^{1+3} M_{13} = M_{13}$ and so on.

**3. Properties of determinants:**

i. The value of a determinant is unchanged, if its rows and columns are interchanged.
ii. Interchanging of any two rows (or columns) will change the sign of the value of the determinant.
iii. If any two rows (or columns) of a determinant are identical, then its value is zero.
iv. If all the elements of any row (or column) are multiplied by a number k, then the value of new determinant so obtained is k times the value of the original determinant.
v. If each element of any row (or column) of a determinant is the sum of two terms, then the determinant can be expressed as the sum of two determinants.
vi. If a constant multiple of all elements of any row (or column) is added to the corresponding elements of any other row (or column), then the value of the new determinant so obtained remains unchanged.

**4. Product of two determinants:**

Let the two determinants of third order be $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$ and $\Delta$ be their product.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

**5. Area of a triangle:**

Area of a triangle whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is given by $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

When the area of the triangle is zero, then the points are collinear.

**6. Solution of non-homogeneous system of linear equations:**

i. The solution of the system of linear equations

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

provided that $D \neq 0$.

**Conditions for consistency:**

a. If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$.

b. If $D = 0$ and $D_1 = D_2 = 0$, then the given system of equations is consistent and has infinitely many solutions.

c. If $D = 0$ and one of $D_1$ and $D_2$ is non-zero, then the given system of equations is inconsistent.

ii. The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$,

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

provided that $D \neq 0$.

**Conditions for consistency:**

a. If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

b. If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations is either consistent with infinitely many solutions or has no solution.

c. If $D = 0$ and at least one of the determinants $D_1$, $D_2$ and $D_3$ is non-zero, then the given system of equations is inconsistent.

**7. Solution of a homogeneous system of linear equations:**

If $a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

is a homogeneous system of equations, such that

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then $x = y = z = 0$ is the only solution and it is known as the trivial solution.

If $D = 0$, then the system is consistent with infinitely many solutions.

## 1.2 Matrices

**1. Matrix:**

A rectangular arrangement of mn numbers (real or complex) in m rows and n columns is called a matrix.

**2. Types of matrices:**

**i. Row matrix:**

A matrix having only one row is called a row-matrix or a row vector.

**ii. Column matrix:**

A matrix having only one column is called a column matrix or a column vector.

**iii. Rectangular matrix:**

A matrix $A = [a_{ij}]_{m\times n}$ is called a rectangular matrix, if number of rows is not equal to number of columns ($m \neq n$).

**iv. Square matrix:**

A matrix $A = [a_{ij}]_{m\times n}$ is called a square matrix, if number of rows is equal to number of columns ($m = n$).

**v. Null matrix or zero matrix:**

A matrix whose all elements are zero is called a null matrix or a zero matrix.

**vi. Diagonal matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is a diagonal matrix, if $a_{ij} = 0 \ \forall \ i \neq j$.

**vii. Scalar matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called a scalar matrix, if all its non-diagonal elements are zero and diagonal elements are same.

**viii. Unit matrix or Identity matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called an identity or unit matrix, if all its non-diagonal elements are zero and diagonal elements are one.

**ix. Triangular matrix:**

A square matrix is said to be triangular matrix if each element above or below the diagonal is zero.

a. A square matrix $A = [a_{ij}]$ is called an upper triangular matrix, if $a_{ij} = 0 \ \forall \ i > j$

b. A square matrix $A = [a_{ij}]$ is called a lower triangular matrix, if $a_{ij} = 0 \ \forall \ i < j$

**x. Singular matrix:**

A square matrix A is called a singular matrix, if $|A| = 0$.

**xi. Non-singular matrix:**

A square matrix A is called a non-singular matrix, if $|A| \neq 0$.

**3. Trace of a matrix:**

The sum of all diagonal elements of a square matrix A is called the trace of matrix A. It is denoted by tr(A).

Thus, $\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$

**4. Submatrix:**

A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a submatrix of the given matrix.

**5. Equality of matrices:**

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if

i. they are of the same order

ii. their corresponding elements are equal.

**6. Algebra of matrices:**

**i. Addition of matrices:**

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices. Then their sum (denoted by A + B) is defined to be matrix $[c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

Similarly, their subtraction A – B is defined as $A - B = [a_{ij} - b_{ij}]_{m \times n} \ \forall$ i, j.

**ii. Multiplication of matrices:**

Let A and B be any two matrices, then their product AB will be defined only when number of columns in A is equal to the number of rows in B. If $A = [a_{ik}]_{m\times n}$ and $B = [b_{kj}]_{n\times p}$, then their product AB is of order $m \times p$ and is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

**7. Transpose of a matrix:**

A matrix obtained from the matrix A by interchanging its rows and columns is called the transpose of A.

Thus, if the order of A is $m \times n$, then the order of $A^T$ is $n \times m$.

**8. Symmetric matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called symmetric matrix, if $A = A^T$ or $a_{ij} = a_{ji} \ \forall$ i, j.

**9. Skew symmetric matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is called skew-symmetric matrix, if $A = -A^T$ or $a_{ij} = -a_{ji} \ \forall$ $i_j$ j.

**10. Orthogonal matrix:**

A square matrix A is called an orthogonal matrix, if $AA^T = A^TA = I$

**11. Idempotent matrix:**

A square matrix A is called an idempotent matrix, if $A^2 = A$.

**12. Nilpotent matrix:**

A square matrix A is said to be a nilpotent matrix of index p, if p is the least positive integer such that $A^p = O$.

**13. Involutory matrix:**

A square matrix A is said to be an involutory matrix, if $A^2 = I$.

**14. Conjugate of a matrix:**

The matrix obtained from a given matrix A by replacing each entry containing complex numbers with its complex conjugate is called conjugate of A..

**15. Hermitian matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be hermitian matrix, if $A = A^\theta$ or $a_{ij} = \bar{a}_{ji}\ \ \forall$ i, j.

**16. Skew – hermitian matrix:**

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be a skew-hermitian matrix if $A = -A^\theta$ or $a_{ij} = -\bar{a}_{ji}\ \ \forall$ i, j

**17. Adjoint of a matrix:**

The adjoint of a square matrix $A = [a_{ij}]$ is the transpose of the matrix of cofactors of elements of A.

Let $A = [a_{ij}]$ be a square matrix and $C_{ij}$ be the cofactor of $a_{ij}$ in A. Then, adj $A = [C_{ij}]^T$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

**18. Inverse of a matrix:**

Let A be a n-rowed square matrix. Then, if there exists a square matrix B of the same order such that AB = I = BA, matrix B is called the inverse of matrix A.

A square matrix A has inverse iff A is non-singular i.e., $A^{-1}$ exists iff $|A| \neq 0$.

The inverse of a non-singular square matrix A is given by $A^{-1} = \frac{1}{|A|}(\text{adj} A)$, if $|A| \neq 0$.

**19. i. Solution of a non-homogeneous system of linear equations:**

If AX = B, then $X = A^{-1} B$ gives a unique solution, provided A is non-singular. But if A is a singular matrix i.e., if $|A| = 0$, then the system of equation AX = B may be consistent with infinitely many solutions or it may be inconsistent.

**Criterion of consistency:**

Let AX = B be a system of n-linear equations in n unknowns.

a. If $|A| \neq 0$, then the system is consistent and has the unique solution given by $X = A^{-1} B$.

b. If $|A| = 0$ and (adj A) B = 0, then the given system of equations is consistent and has infinitely many solutions.

c. If $|A| = 0$ and (adj A) B $\neq 0$, then the given system of equations is inconsistent.

**ii. Solution of a homogeneous system of linear equations:**

Let AX = O be a homogeneous system of n-linear equations with n-unknowns. If A is a non-singular matrix, then the system of equations has a unique solution X = O i.e., $x_1 = x_2 = \ldots = x_n = 0$.

This solution is known as a trivial solution. A system AX = O of n homogeneous linear equations in n unknowns, has non-trivial solution iff the coefficient matrix A is singular.

**Shortcuts**

1. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

2. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

3. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

4. $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a - b)(b - c)(c - a)$

5. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$, where $A_1, B_1, C_1$ are co-factors of $a_1, b_1, c_1$, etc. then $\Delta' = \Delta^2$.

6. If a square matrix A is orthogonal i.e., if $A^TA = I$, then det A is 1 or –1.

7. If A is an involutory matrix, then $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).\ \frac{1}{2}(I - A) = 0$.

8. If a square matrix A is unitary i.e., if $A^{\theta}A = I$, then |det A| = 1.

9. i. If A is a square matrix of order 2, then |adj A| = |A|.
   ii. If A is a square matrix of order 3, then $|\text{adj } A| = |A|^2$.

10. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{(ad - bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $(ad - bc \neq 0)$

11. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

12. If $A^m = I$ for some positive integer m, where A is a square matrix, then A is invertible and $A^{-1} = A^{m-1}$.

## Multiple Choice Questions

### 1.1 Determinants

#### 1.1.1 Determinants of order two and three, properties and evaluation of determinants

1. The value of the determinant $\begin{vmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{vmatrix}$ is

(A) $-75$ (B) $25$
(C) $0$ (D) $-25$

2. $\begin{vmatrix} 19 & 17 & 15 \\ 9 & 8 & 7 \\ 1 & 1 & 1 \end{vmatrix} =$

(A) $0$ (B) $187$
(C) $354$ (D) $54$

3. $\begin{vmatrix} \frac{1}{a} & 1 & bc \\ \frac{1}{b} & 1 & ca \\ \frac{1}{c} & 1 & ab \end{vmatrix} =$ **[RPET 2002]**

(A) $0$ (B) abc
(C) $\frac{1}{abc}$ (D) None of these

4. $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} =$ **[MP PET 1991]**

(A) $1 + a^2 + b^2 + c^2$ (B) $1 - a^2 + b^2 + c^2$
(C) $1 + a^2 + b^2 - c^2$ (D) $1 + a^2 - b^2 + c^2$

5. If $\begin{vmatrix} 1 & 2 & 3 \\ 2 & x & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0$, then $x =$

**[Karnataka CET 1994]**

(A) $-\frac{5}{2}$ (B) $-\frac{2}{5}$
(C) $\frac{5}{2}$ (D) $\frac{2}{5}$

6. What is value of $x$, if $\begin{vmatrix} 8 & -5 & 1 \\ 5 & x & 1 \\ 6 & 3 & 1 \end{vmatrix} = 2$?

(A) $4$ (B) $8$
(C) $5$ (D) $9$

7. If $\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -1 \end{vmatrix} = 0$, then the value of k is

**[IIT 1979]**

(A) $-1$ (B) $0$
(C) $1$ (D) None of these

8. The value of $x$, if $\begin{vmatrix} -x & 1 & 0 \\ 1 & -x & 1 \\ 0 & 1 & -x \end{vmatrix} = 0$, is equal to **[Pb. CET 2002]**

(A) $\pm\sqrt{6}$ (B) $\pm\sqrt{2}$
(C) $\pm\sqrt{3}$ (D) $\sqrt{2}, \sqrt{3}$

9. If $\Delta_1 = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & 0 \\ c & d \end{vmatrix}$, then $\Delta_2\Delta_1$ is equal to **[RPET 1984]**

(A) ac (B) bd
(C) $(b - a)(d - c)$ (D) None of these

10. If $\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}^2 = \begin{vmatrix} 3 & 2 \\ 1 & x \end{vmatrix} - \begin{vmatrix} x & 3 \\ -2 & 1 \end{vmatrix}$, then $x =$

**[RPET 1996]**

(A) $-14$ (B) $2$
(C) $6$ (D) $7$

11. $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} =$

**[Tamilnadu (Engg.) 2002]**

(A) $7$ (B) $10$
(C) $13$ (D) $17$

12. The minors of $-4$ and $9$ and the co-factors of $-4$ and $9$ in determinant $\begin{vmatrix} -1 & -2 & 3 \\ -4 & -5 & -6 \\ -7 & 8 & 9 \end{vmatrix}$ are respectively **[J & K 2005]**

(A) $42, 3; -42, 3$ (B) $-42, -3; 42, -3$
(C) $42, 3; -42, -3$ (D) $42, 3; 42, 3$

13. If $A = \begin{vmatrix} 5 & 6 & 3 \\ -4 & 3 & 2 \\ -4 & -7 & 3 \end{vmatrix}$, then cofactors of the elements of 2nd row are **[RPET 2002]**

(A) $39, -3, 11$ (B) $-39, 3, 11$
(C) $-39, 27, 11$ (D) $-39, -3, 11$

14. $\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} =$ **[RPET 1989]**

(A) 0 (B) $ma_1a_2a_3$
(C) $ma_1a_2b_3$ (D) $mb_1a_2a_3$

15. The value of $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is

(A) $5^2$ (B) 0
(C) $5^{13}$ (D) $5^9$

16. If $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$, then $\begin{vmatrix} x & 2y & z \\ 2p & 4q & 2r \\ a & 2b & c \end{vmatrix}$ equals **[RPET 1999]**

(A) $\Delta^2$ (B) $4\Delta$
(C) $3\Delta$ (D) None of these

17. If $\begin{vmatrix} a & b & c \\ m & n & p \\ x & y & z \end{vmatrix} = k$, then $\begin{vmatrix} 6a & 2b & 2c \\ 3m & n & p \\ 3x & y & z \end{vmatrix} =$ **[Tamilnadu (Engg.) 2002]**

(A) $\frac{k}{6}$ (B) 2k
(C) 3k (D) 6k

18. If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix} =$ **[RPET 1986]**

(A) $\Delta$ (B) $k\Delta$
(C) $3k\Delta$ (D) $k^3\Delta$

19. $\begin{vmatrix} 1 & 5 & \pi \\ \log_e e & 5 & \sqrt{5} \\ \log_{10} 10 & 5 & e \end{vmatrix} =$

(A) $\sqrt{\pi}$ (B) e
(C) 1 (D) 0

20. $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix} =$ **[RPET 1990, 99]**

(A) abc (B) 1/abc
(C) ab + bc + ca (D) 0

21. The value of the determinant $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ is equal to **[Roorkee 1992]**

(A) –4 (B) 0
(C) 1 (D) 4

22. $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \cos^2 x & \sin^2 x & 1 \\ -10 & 12 & 2 \end{vmatrix} =$ **[EAMCET 1994]**

(A) 0
(B) $12\cos^2 x - 10\sin^2 x$
(C) $12\sin^2 x - 10\cos^2 x - 2$
(D) $10\sin 2x$

23. $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix} =$ **[MP PET 1996]**

(A) 0 (B) –39
(C) 96 (D) 57

24. The value of $\begin{vmatrix} 41 & 42 & 43 \\ 44 & 45 & 46 \\ 47 & 48 & 49 \end{vmatrix} =$ **[Karnataka CET 2001]**

(A) 2 (B) 4
(C) 0 (D) 1

25. The determinant $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$ is equal to

(A) 0 (B) 1
(C) –1 (D) none of these

26. If $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$, then **[Tamilnadu (Engg.) 2002]**

(A) B = 4A (B) B = –4A
(C) B = –A (D) B = 6A

27. $\begin{vmatrix} x & 4 & y+z \\ y & 4 & z+x \\ z & 4 & x+y \end{vmatrix} =$ **[Karnataka CET 1991]**

(A) 4 (B) $x + y + z$
(C) $xyz$ (D) 0

28. If a ≠ 6, b, c satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then abc =

**[EAMCET 2000]**

(A) a + b + c (B) 0
(C) $b^3$ (D) ab + bc

29. If ω is a complex cube root of unity, then the determinant $\begin{vmatrix} 2 & 2\omega & -\omega^2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$

(A) 0 (B) 1
(C) –1 (D) None of these

30. If ω is a complex cube root of unity, then $\begin{vmatrix} 1 & \omega & -\omega^2/2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} =$

(A) 0 (B) 1
(C) ω (D) $\omega^2$

31. The value of the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is

**[MP PET 1993; Karnataka CET 1994; Pb. CET 2004]**

(A) a + b + c (B) $(a + b + c)^2$
(C) 0 (D) 1 + a + b + c

32. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} =$ **[RPET 1996]**

(A) 1 (B) 0
(C) $x$ (D) $xy$

33. The roots of the equation $\begin{vmatrix} 0 & x & 16 \\ x & 5 & 7 \\ 0 & 9 & x \end{vmatrix} = 0$ are

**[Pb. CET 2001; Karnataka CET 1994]**

(A) 0, 12, 12 (B) 0, 12, –12
(C) 0, 12, 16 (D) 0, 9, 16

34. The roots of the determinant (in $x$)
$\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0$ are

**[EAMCET 1993]**

(A) $x$ = a, b (B) $x$ = –a, –b
(C) $x$ = –a, b (D) $x$ = a, –b

35. The roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ are

**[IIT 1987; MP PET 2002]**

(A) –1, –2 (B) –1, 2
(C) 1, –2 (D) 1, 2

36. If $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$, then $x$ =

**[MP PET 1991]**

(A) 1, 9 (B) –1, 9
(C) –1, –9 (D) 1, –9

37. If $\begin{vmatrix} x+1 & 1 & 1 \\ 2 & x+2 & 2 \\ 3 & 3 & x+3 \end{vmatrix} = 0$, then $x$ is

**[Kerala (Engg.) 2002]**

(A) 0, –6 (B) 0, 6
(C) 6 (D) –6

38. Solution of the equation $\begin{vmatrix} 1 & 1 & x \\ p+1 & p+1 & p+x \\ 3 & x+1 & x+2 \end{vmatrix} = 0$ are **[AMU 2002]**

(A) $x$ = 1, 2
(B) $x$ = 2, 3
(C) $x$ = 1, p, 2
(D) $x$ = 1, 2, –p

39. A root of the equation
$\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

**[Roorkee 1991; RPET 2001; J & K 2005]**

(A) 6 (B) 3
(C) 0 (D) None of these

40. The roots of the equation
$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0$ are

**[MP PET 1989; Roorkee 1998]**

(A) 0, –3
(B) 0, 0, –3
(C) 0, 0, 0, –3
(D) None of these

41. The roots of the equation $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ are **[Karnataka CET 1992]**
(A) 1, 2 (B) –1, 2
(C) 1, –2 (D) –1, –2

42. One of the roots of the given equation $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ is **[MP PET 1988, 2002; RPET 1996]**
(A) –(a + b) (B) –(b + c)
(C) –a (D) –(a + b + c)

43. If a ≠ b ≠ c, the value of $x$ which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ is **[EAMCET 1988; Karnataka CET 1991; MNR 1980; MP PET 1988, 99, 2001; DCE 2001]**
(A) $x = 0$ (B) $x = a$
(C) $x = b$ (D) $x = c$

44. If $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$, then the values of $x$ are **[RPET 1997]**
(A) $0, \frac{2}{3}$ (B) $\frac{2}{3}, \frac{11}{3}$
(C) $\frac{1}{2}, 1$ (D) $\frac{11}{3}, 1$

45. If –9 is a root of the equation $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then the other two roots are **[IIT 1983; MNR 1992; MP PET 1995; DCE 1997; UPSEAT 2001]**
(A) 2, 7 (B) –2, 7
(C) 2, –7 (D) –2, –7

46. $\begin{vmatrix} 1 & 1 & 1 \\ ^nC_1 & ^{n+1}C_1 & ^{n+2}C_1 \\ ^nC_2 & ^{n+1}C_2 & ^{n+2}C_2 \end{vmatrix} =$
(A) 0 (B) 1
(C) –1 (D) none of these

47. The value of the determinant $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$ is
(A) $(x-y)(y-z)(z-x)$
(B) $2(x-y)(y-z)(z-x)$
(C) $(x-y)(y-z)(z-x)(x+y+z)$
(D) none of these

48. The value of the determinant $\begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$ is **[Orissa JEE 2003]**
(A) 2 (10! 11!)
(B) 2 (10! 13!)
(C) 2 (10! 11! 12!)
(D) 2 (11! 12! 13!)

49. The value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b+c-a & c+a-b & a+b-c \end{vmatrix}$ is **[RPET 1986]**
(A) abc (B) a + b + c
(C) ab + bc + ca (D) None of these

50. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} =$ **[IIT 1988; MP PET 1990, 91; RPET 2002]**
(A) 0
(B) $a^3+b^3+c^3-3abc$
(C) 3abc
(D) $(a+b+c)^3$

51. The determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$ is not equal to **[MP PET 1988]**
(A) $\begin{vmatrix} 2 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 3 & 6 \end{vmatrix}$ (B) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \end{vmatrix}$
(C) $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 3 \\ 1 & 9 & 6 \end{vmatrix}$ (D) $\begin{vmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 10 & 3 & 6 \end{vmatrix}$

52. The value of $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix}$ is **[Karnataka CET 2004]**
(A) 1
(B) 0
(C) (a – b) (b – c) (c – a)
(D) (a + b) (b + c) (c + a)

53. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} =$ **[RPET 1990, 95]**
(A) $(a+b+c)^2$
(B) $(a+b+c)^3$
(C) (a + b + c) (ab + bc + ca)
(D) None of these

54. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} =$ **[MP PET 1990]**
(A) $a^3+b^3+c^3-3abc$
(B) $3abc-a^3-b^3-c^3$
(C) $a^3+b^3+c^3-a^2b-b^2c-c^2a$
(D) $(a+b+c)(a^2+b^2+c^2+ab+bc+ca)$

55. $\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix} =$ **[IIT 1986; MNR 1985; MP PET 1998; Pb. CET 2003]**
(A) $a^2+b^2+c^2-3abc$ (B) 3ab
(C) 3a + 5b (D) 0

56. $\begin{vmatrix} a-1 & a & bc \\ b-1 & b & ca \\ c-1 & c & ab \end{vmatrix} =$ **[RPET 1988]**
(A) 0
(B) –(a – b) (b – c) (c – a)
(C) $a^3+b^3+c^3-3abc$
(D) None of these

57. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} =$ **[IIT 1980]**
(A) abc (B) 4abc
(C) $4a^2b^2c^2$ (D) $a^2b^2c^2$

58. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} =$ **[RPET 1992; Kerala (Engg.) 2002]**
(A) $xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$
(B) $xyz$
(C) $1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$
(D) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

59. If $a^{-1}+b^{-1}+c^{-1}=0$ such that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = \lambda$, then the value of λ is **[RPET 2000]**
(A) 0 (B) abc
(C) –abc (D) None of these

60. If ω is a cube root of unity, then $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} =$ **[MNR 1990; MP PET 1999]**
(A) $x^3+1$ (B) $x^3+\omega$
(C) $x^3+\omega^2$ (D) $x^3$

61. If $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = k(x+y+z)(x-z)^2$, then k =
(A) $2xyz$ (B) 1
(C) $xyz$ (D) $x^2y^2z^2$

62. The value of $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ is equal to **[RPET 1989]**
(A) 0 (B) 679
(C) 779 (D) 1000

63. If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax-12$, then the value of A is **[IIT 1982]**
(A) 12 (B) 24
(C) –12 (D) –24

64. If a, b, c are positive integers, then the determinant $\Delta = \begin{vmatrix} a^2+x & ab & ac \\ ab & b^2+x & bc \\ ac & bc & c^2+x \end{vmatrix}$ is divisible by
(A) $x^3$ (B) $x^2$
(C) $a^2+b^2+c^2$ (D) None of these

65. If $D_p = \begin{vmatrix} p & 15 & 8 \\ p^2 & 35 & 9 \\ p^3 & 25 & 10 \end{vmatrix}$, then $D_1 + D_2 + D_3 + D_4 + D_5 =$
**[Kurukshetra CEE 1998]**
(A) 0 (B) 25
(C) 625 (D) None of these

66. $2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ac & c^2-ab \end{vmatrix} =$
**[EAMCET 1991; UPSEAT 1999]**
(A) 0 (B) 1
(C) 2 (D) 3abc

67. The value of $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$ is equal to
**[Kerala (Engg.) 2001]**
(A) $9a^2(a+b)$ (B) $9b^2(a+b)$
(C) $a^2(a+b)$ (D) $b^2(a+b)$

68. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} =$
**[UPSEAT 2002; AMU 2005]**
(A) 0 (B) 2abc
(C) $a^2b^2c^2$ (D) None of these

69. If $\begin{vmatrix} 1+\sin^2\theta & \sin^2\theta & \sin^2\theta \\ \cos^2\theta & 1+\cos^2\theta & \cos^2\theta \\ 4\sin 4\theta & 4\sin 4\theta & 1+4\sin 4\theta \end{vmatrix} = 0$, then $\sin 4\theta$ is equal to **[Orissa JEE 2005]**
(A) $\frac{1}{2}$ (B) 1
(C) $-\frac{1}{2}$ (D) $-1$

70. $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix} =$
**[MNR 1985; UPSEAT 2000]**
(A) 2 (B) $-2$
(C) $x^2-2$ (D) None of these

71. If $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix}$, where $a = i$, $b = \omega$, $c = \omega^2$, then $\Delta$ is equal to
(A) i (B) $-\omega^2$
(C) $\omega$ (D) $-i$

72. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$, then f(100) is equal to
**[IIT 1999, DCE 2005]**
(A) 0 (B) 1
(C) 100 (D) $-100$

73. The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is
(A) k (a + b) (b + c) (c + a)
(B) $k(a^2+b^2+c^2)$
(C) k (a – b) (b – c) (c – a)
(D) k (a + b – c) (b + c – a) (c + a – b)

74. If $D_1 = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then
(A) $D_1 = 2D_2$ (B) $D_2 = 2D_1$
(C) $D_1 = D_2$ (D) $D_1 \neq D_2$

75. If $a \neq p$, $b \neq q$, $c \neq 0$ and $\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix} = 0$, then $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} =$
**[EAMCET 2003]**
(A) 3 (B) 2
(C) 1 (D) 0

76. If a, b, c are unequal, what is the condition that the value of the following determinant is zero

$$\Delta = \begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix}$$

**[IIT 1985; DCE 1999]**

(A) $1 + abc = 0$
(B) $a + b + c + 1 = 0$
(C) $(a - b)(b - c)(c - a) = 0$
(D) None of these

77. Let a, b, c be positive and not all equal, the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, is

**[DCE 2006]**

(A) positive (B) negative
(C) zero (D) none of these

78. If ab + bc + ca = 0 and $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$, then one of the value of $x$ is

**[AMU 2000]**

(A) $(a^2 + b^2 + c^2)^{\frac{1}{2}}$
(B) $\left[\frac{3}{2}\left(a^2+b^2+c^2\right)\right]^{\frac{1}{2}}$
(C) $\left[\frac{1}{2}\left(a^2+b^2+c^2\right)\right]^{\frac{1}{2}}$
(D) None of these

79. If $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$

$$= \begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & 2-\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$$, the value of t is

**[IIT 1981]**

(A) 16 (B) 18
(C) 17 (D) 19

80. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$, then

f($x$) is a polynomial of degree **[AIEEE 2005]**

(A) 2 (B) 3
(C) 0 (D) 1

81. The determinant

$$\Delta = \begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$ is

equal to

(A) $\cos\alpha \cos\beta \cos\gamma$
(B) $\cos\alpha + \cos\beta + \cos\gamma$
(C) 1
(D) 0

82. The value of the determinant

$$\Delta = \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$$ is

(A) independent of $\alpha$
(B) independent of $\beta$
(C) independent of $\alpha$ and $\beta$
(D) none of these

83. $\begin{vmatrix} 1 & 1 & 1 \\ \cos(nx) & \cos(n+1)x & \cos(n+2)x \\ \sin(nx) & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$ does not depend

**[RPET 2000]**

(A) on $x$ (B) on n
(C) on both $x$ and n (D) none of these

84. If A, B, C be the angles of a triangle, then

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$ is

**[Karnataka CET 2002]**

(A) 1
(B) 0
(C) cos A cos B cos C
(D) cos A + cos B cos C

85. The value of the determinant

$$\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos\alpha \\ \cos(\alpha-\beta) & 1 & \cos\beta \\ \cos\alpha & \cos\beta & 1 \end{vmatrix}$$ is

**[UPSEAT 2003]**

(A) $\alpha^2 + \beta^2$ (B) $\alpha^2 - \beta^2$
(C) 1 (D) 0

86. If $A = \begin{vmatrix} \sin(\theta+\alpha) & \cos(\theta+\alpha) & 1 \\ \sin(\theta+\beta) & \cos(\theta+\beta) & 1 \\ \sin(\theta+\gamma) & \cos(\theta+\gamma) & 1 \end{vmatrix}$, then

**[Orissa JEE 2003]**

(A) $A = 0$ for all $\theta$
(B) A is an odd function of $\theta$
(C) $A = 0$ for $\theta = \alpha + \beta + \gamma$
(D) A is independent of $\theta$

87. If $x$ is a positive integer, then $\begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$ is equal to

**[DCE 2009]**

(A) $2\,x!\,(x+1)!$
(B) $2\,x!\,(x+1)!\,(x+2)!$
(C) $2\,x!\,(x+3)!$
(D) $2\,(x+1)!\,(x+2)!\,(x+3)!$

88. Let a, b, c be such that b (a + c) ≠ 0. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then the value of n is

**[AIEEE 2009]**

(A) zero
(B) any even integer
(C) any odd integer
(D) any integer

89. If $f(x) = \begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$, then

f(1).f(3) + f(3).f(5) + f(5).f(1) =

**[Kerala (Engg.) 2005]**

(A) f(1) (B) f(3)
(C) f(1) + f(3) (D) f(1) + f(5)

90. If $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,abc(a+b+c)^3$,

then the value of k is

**[Tamilnadu (Engg.) 2001]**

(A) –1 (B) 1
(C) 2 (D) –2

91. If a, b, c are non-zero complex numbers satisfying $a^2 + b^2 + c^2 = 0$ and

$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = ka^2b^2c^2$, then k is equal to

**[AIEEE 2012]**

(A) 3 (B) 2
(C) 4 (D) 1

## 1.1.2 Area of a triangle using determinants

92. If the vertices of a triangle are A(5, 4), B(–2, 4) and C(2, –6), then its area in sq. units is
(A) 70 sq. units (B) 38 sq. units
(C) 30 sq. units (D) 35 sq. units

93. If the points ($x$, –2), (5, 2), (8, 8) are collinear, then $x$ is equal to
(A) –3 (B) $\frac{1}{3}$
(C) 1 (D) 3

94. If the area of a triangle is 4 square units whose vertices are (k, 0), (4, 0), (0, 2), then the values of k are
(A) 0, 8 (B) 4, 3
(C) 2, 4 (D) 2, 6

95. If the points (2, –3), (λ, –1) and (0, 4) are collinear, then the value of λ is
(A) $\frac{4}{7}$ (B) 2
(C) $\frac{10}{7}$ (D) $\frac{7}{10}$

96. If the points (a, 0), (0, b) and (1, 1) are collinear, then $\frac{1}{a}+\frac{1}{b}$ is equal to
(A) a (B) 1
(C) – 1 (D) b

97. The equation of the line joining the points A (1, 2) and B (3, 6) is
(A) $y = 2x$ (B) $x - y = 0$
(C) $2y = x$ (D) $x + y = 0$

98. If the points $(a_1, b_1)$, $(a_2, b_2)$ and $(a_1 + a_2, b_1 + b_2)$ are collinear, then
(A) $a_1b_2 = a_2b_1$ (B) $a_1b_2 = -a_2b_1$
(C) $a_1b_1 = a_2b_2$ (D) $a_1b_1 + a_2b_2 = 0$

99. Points (k, 2 – 2k), (–k + 1, 2k) and (–4 – k, 6 – 2k) are collinear, if k is equal to
(A) 2, –1 (B) –1, 3
(C) 1, –2 (D) $-1, \frac{1}{2}$

100. If $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$ are vertices of an equilateral triangle whose each side is equal to a, then $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$ is

(A) $\frac{\sqrt{3}}{4}a^2$ (B) $3a^4$
(C) $3a^2$ (D) $\frac{3a^4}{16}$

101. A triangle has its three sides equal to a, b and c. If the co-ordinates of its vertices are $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$, then $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2$ is

(A) $(a + b + c)(b + c - a)(c + a - b)(a + b - c)$
(B) $(a - b + c)^2 (b + c - a) (a + b - c)$
(C) $(a - b - c)(b + c - a)(c - a - b)(a + b - c)$
(D) $(a + b - c)(b + c - a)(c - a + b)(a - b + c)$

### 1.1.3 Test of consistency and solution of simultaneous linear equations in two or three variables

102. If $2x + 3y - 5z = 7, x + y + z = 6,$ $3x - 4y + 2z = 1$, then $x =$

(A) $\begin{vmatrix} 2 & -5 & 7 \\ 1 & 1 & 6 \\ 3 & 2 & 1 \end{vmatrix} \div \begin{vmatrix} 7 & 3 & -5 \\ 6 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix}$

(B) $\begin{vmatrix} -7 & 3 & -5 \\ -6 & 1 & 1 \\ -1 & -4 & 2 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & 1 \\ 3 & -4 & 2 \end{vmatrix}$

(C) $\begin{vmatrix} 7 & 3 & -5 \\ 6 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix} \div \begin{vmatrix} 2 & 3 & -5 \\ 1 & 1 & 1 \\ 3 & -4 & 2 \end{vmatrix}$

(D) None of these

103. The number of solutions of the system of equations $2x + y - z = 7, x - 3y + 2z = 1, x + 4y - 3z = 5$ is **[EAMCET 2003]**

(A) 3 (B) 2
(C) 1 (D) 0

104. The value of $\lambda$ for which the system of equations $2x - y - z = 12$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ has no solution is **[IIT Screening 2004]**

(A) 3 (B) $-3$
(C) 2 (D) $-2$

105. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution, if **[EAMCET 1994; DCE 2000]**

(A) $k \neq 0$ (B) $-1 < k < 1$
(C) $-2 < k < 2$ (D) $k = 0$

106. Let a, b, c be positive real numbers. The following system of equations
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,
$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has

(A) no solution
(B) unique solution
(C) infinitely many solutions
(D) none of these

107. If the system of equations $x + 2y - 3z = 1$, $(k + 3)z = 3$, $(2k + 1)x + z = 0$ is inconsistent, then the value of k is **[Roorkee 2000]**

(A) $-3$ (B) $\frac{1}{2}$
(C) 0 (D) 2

108. The system of equations
$\alpha x + y + z = \alpha - 1$
$x + \alpha y + z = \alpha - 1$
$x + y + \alpha z = \alpha - 1$
has no solution, if $\alpha$ is **[AIEEE 2005]**

(A) not equal to $-2$ (B) 1
(C) $-2$ (D) either $-2$ or 1

109. The system of equations $x_1 - x_2 + x_3 = 2$, $3x_1 - x_2 + 2x_3 = -6$ and $3x_1 + x_2 + x_3 = -18$ has **[AMU 2001]**

(A) no solution
(B) exactly one solution
(C) infinite solutions
(D) none of these

110. If $a_1x + b_1y + c_1z = 0, a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$ and $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the given system has **[Roorkee 1990]**

(A) one trivial and one non-trivial solution
(B) no solution
(C) one solution
(D) infinite solutions

111. The following system of equations $3x - 2y + z = 0$, $\lambda x - 14y + 15z = 0$, $x + 2y - 3z = 0$ has a solution other than $x = y = z = 0$ for $\lambda$ equal to **[MP PET 1990]**
(A) 1 (B) 2
(C) 3 (D) 5

112. $x + ky - z = 0$, $3x - ky - z = 0$ and $x - 3y + z = 0$ has a non-zero solution for k = **[IIT 1988]**
(A) –1 (B) 0
(C) 1 (D) 2

113. The number of solutions of the equations $x + y - z = 0$, $3x - y - z = 0$, $x - 3y + z = 0$ is **[MP PET 1992]**
(A) 0 (B) 1
(C) 2 (D) Infinite

114. If $x + y - z = 0$, $3x - \alpha y - 3z = 0$, $x - 3y + z = 0$ has a non – zero solution, then $\alpha$ = **[MP PET 1990]**
(A) –1 (B) 0
(C) 1 (D) –3

115. The number of solutions of the equations $x + 4y - z = 0$, $3x - 4y - z = 0$, $x - 3y + z = 0$ is **[MP PET 1992]**
(A) 0 (B) 1
(C) 2 (D) Infinite

116. The value of a for which the system of equations $a^3x + (a + 1)^3 y + (a + 2)^3 z = 0$, $ax + (a + 1) y + (a + 2) z = 0$, $x + y + z = 0$, has a non-zero solution is **[Pb. CET 2000]**
(A) –1 (B) 0
(C) 1 (D) None of these

117. The value of k for which the system of equations $x + ky + 3z = 0$, $3x + ky - 2z = 0$, $2x + 3y - 4z = 0$ has a non-trivial solution is
(A) 15 (B) $\frac{31}{2}$
(C) 16 (D) $\frac{33}{2}$

118. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$ and $x + y - z = 0$ has a non zero solution, then the possible values of k are **[IIT Screening 2000]**
(A) –1, 2 (B) 1, 2
(C) 0, 1 (D) –1, 1

119. Set of equations $a + b - 2c = 0$, $2a - 3b + c = 0$ and $a - 5b + 4c = \alpha$ is consistent for $\alpha$ equal to **[Orissa JEE 2004]**
(A) 1 (B) 0
(C) –1 (D) 2

120. If the system of equations $3x - 2y + z = 0$, $\lambda x - 14y + 15z = 0$, $x + 2y + 3z = 0$ have a non-trivial solution, then $\lambda$ = **[EAMCET 1993]**
(A) 5 (B) –5
(C) –29 (D) 29

121. For what value of $\lambda$, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = 12$ is inconsistent? **[AIEEE 2002]**
(A) $\lambda = 1$ (B) $\lambda = 2$
(C) $\lambda = -2$ (D) $\lambda = 3$

122. If the system of equations $x + ay = 0$, $az + y = 0$ and $ax + z = 0$ has infinite solutions, then the value of a is **[IIT Screening 2003]**
(A) –1 (B) 1
(C) 0 (D) No real values

123. Value of $\lambda$ for which the homogeneous system of equations $2x + 3y - 2z = 0$, $2x - y + 3z = 0$, $7x + \lambda y - z = 0$ has non-trivial solutions is
(A) $\frac{57}{10}$ (B) $\frac{-57}{10}$
(C) $\frac{81}{10}$ (D) $\frac{55}{10}$

124. The system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0$, $-x - y + \lambda z = 0$, will have a non-zero solution if real values of $\lambda$ are given by **[IIT 1984]**
(A) 0 (B) 1
(C) 3 (D) $\sqrt{3}$

125. Let the homogeneous system of linear equations
$px + y + z = 0$, $x + qy + z = 0$, $x + y + rz = 0$, where p, q, r $\neq$ 1, have a non-zero solution, then the value of $\frac{1}{1-p} + \frac{1}{1-q} + \frac{1}{1-r}$ is
(A) –1 (B) 0
(C) 2 (D) 1

126. If $f(x) = ax^2 + bx + c$ is a quadratic function such that f(1) = 8, f(2) = 11 and f(–3) = 6, then f(0) is equal to
(A) 0 (B) 6
(C) 8 (D) 11

348. If $D_r = \begin{vmatrix} r & 1 & \frac{n(n+1)}{2} \\ 2r-1 & 4 & n^2 \\ 2^{r-1} & 5 & 2^n - 1 \end{vmatrix}$, then the value of $\sum_{r=1}^{n} D_r$ is **[DCE 2006]**

(A) 0 (B) 1
(C) $\frac{n(n+1)(2n+1)}{6}$ (D) none of these

349. If A is a square matrix of order n and $A = kB$, where k is a scalar, then $|A|=$ **[Karnataka CET 1992]**

(A) $|B|$ (B) $k|B|$
(C) $k^n|B|$ (D) $n|B|$

350. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB| =$ **[Karnataka CET 2000]**

(A) $-9$ (B) $-81$ (C) $-27$ (D) 81

351. If $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix} X = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$, then X = **[MP PET 1994]**

(A) $\begin{bmatrix} -3 & 4 \\ 14 & -13 \end{bmatrix}$ (B) $\begin{bmatrix} 3 & -4 \\ -14 & 13 \end{bmatrix}$
(C) $\begin{bmatrix} 3 & 4 \\ 14 & 13 \end{bmatrix}$ (D) $\begin{bmatrix} -3 & 4 \\ -14 & 13 \end{bmatrix}$

352. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then which of the following statements is not correct? **[DCE 2001]**

(A) A is orthogonal matrix
(B) $A'$ is orthogonal matrix
(C) Determinant A = 1
(D) A is not invertible

353. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that A = BX, then X is equal to **[DCE 1995]**

(A) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (B) $\frac{1}{2}\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$
(C) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ (D) none of these

354. If AX = B, $B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{bmatrix}$, then X is equal to

(A) $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ (B) $\begin{bmatrix} 1/2 \\ -1/2 \\ 2 \end{bmatrix}$
(C) $\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$ (D) $\begin{bmatrix} 3 \\ 3/4 \\ -3/4 \end{bmatrix}$

355. For each real number $x$ such that $-1 < x < 1$, let $A(x)$ be the matrix $(1-x)^{-1}\begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$ and $z = \frac{x+y}{1+xy}$. Then,

(A) $A(z) = A(x) + A(y)$
(B) $A(z) = A(x)[A(y)]^{-1}$
(C) $A(z) = A(x)A(y)$
(D) $A(z) = A(x) - A(y)$

356. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. If AX = B, then X =

(A) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (B) $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$
(C) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ (D) $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

357. Let $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. The only correct statement about the matrix A is **[AIEEE 2004]**

(A) $A^2 = I$
(B) $A = (-1)I$, where I is a unit matrix
(C) $A^{-1}$ does not exist
(D) A is a zero matrix

358. If [ ] denotes the greatest integer less than or equal to the real number under consideration, and $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, then the value of the determinant

$$\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$$, is **[DCE 1998]**

(A) [z] (B) [y]
(C) [x] (D) none of these

359. If $C = 2\cos\theta$, then the value of the determinant

$$\Delta = \begin{vmatrix} C & 1 & 0 \\ 1 & C & 1 \\ 0 & 1 & C \end{vmatrix}$$, is **[Orissa JEE 2002]**

(A) $\frac{\sin 4\theta}{\sin\theta}$
(B) $\frac{2\sin^2 2\theta}{\sin\theta}$
(C) $4\cos^2\theta(2\cos\theta - 1)$
(D) none of these

360. Let $P = [a_{ij}]$ be a $3 \times 3$ matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is **[IIT 2012]**

(A) $2^{10}$ (B) $2^{11}$ (C) $2^{12}$ (D) $2^{13}$

361. If $\begin{vmatrix} a & b & a\alpha - b \\ b & c & b\alpha - c \\ 2 & 1 & 0 \end{vmatrix} = 0$ and $\alpha \ne \frac{1}{2}$, then

(A) a, b, c are in A. P.
(B) a, b, c are in G.P.
(C) a, b, c are in H. P.
(D) none of these

362. Consider the system of linear equations $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, and $a_3x + b_3y + c_3z + d_3 = 0$. Let us denote by $\Delta(abc)$ the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ if $\Delta(abc) \ne 0$, then the value of $x$ in the unique solution of the above equations is **[Pb. CET 2004]**

(A) $\frac{\Delta(bcd)}{\Delta(abc)}$ (B) $\frac{-\Delta(bcd)}{\Delta(abc)}$
(C) $\frac{\Delta(acd)}{\Delta(abc)}$ (D) $-\frac{\Delta(abd)}{\Delta(abc)}$

363. Let M be a $3 \times 3$ matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}.$$

Then the sum of the diagonal entries of M is **[IIT 2011]**

(A) 7 (B) 8
(C) 9 (D) 6

364. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^TQ^{2005}P$ is equal to **[IIT Screening 2005]**

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} \sqrt{3}/2 & 2005 \\ 1 & 0 \end{bmatrix}$
(C) $\begin{bmatrix} 1 & 2005 \\ \sqrt{3}/2 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & \sqrt{3}/2 \\ 0 & 2005 \end{bmatrix}$

365. If P is a $3 \times 3$ matrix such that $P^T = 2P + I$, where $P^T$ is the transpose of P and I is $3 \times 3$ identity, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ x \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that PX = **[IIT 2012]**

(A) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) X
(C) 2X (D) −X

366. Let A and B be $3 \times 3$ matrices of real numbers, where A is symmetric, B is skew-symmetric and $(A + B)(A - B) = (A - B)(A + B)$ if $(AB)^T = (-1)^n AB$, then

(A) $n \in Z$
(B) $n \in N$
(C) n is an even natural number
(D) n is an odd natural number

367. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$, then $|\alpha|$ equals **[AIEEE 2007]**

(A) $\frac{1}{5}$ (B) 5
(C) $5^2$ (D) 1

368. If f(α) = $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix}$, then f $\left(\sqrt[3]{3}\right)$ is equal to **[Kerala PET 2008]**

(A) 1 (B) –4
(C) 4 (D) 2

369. If the matrix

$M_r = \begin{bmatrix} r & r-1 \\ r-1 & r \end{bmatrix}$, r = 1, 2, 3, ...., then the value of det $(M_1)$ + det$(M_2)$ +....+ det $(M_{2008})$ is **[Kerala PET 2008]**

(A) 2007
(B) 2008
(C) $(2008)^2$
(D) $(2007)^2$

370. If A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds for all n ≥ 1, by the principle of mathematical induction? **[AIEEE 2005]**

(A) $A^n = 2^{n-1} A + (n-1)I$
(B) $A^n = nA + (n-1)I$
(C) $A^n = 2^{n-1} A - (n-1)I$
(D) $A^n = nA - (n-1)I$

371. If $\alpha^3 \neq 1$ and $\alpha^9 = 1$, then the value of

$\begin{vmatrix} \alpha & \alpha^3 & \alpha^5 \\ \alpha^3 & \alpha^5 & \alpha \\ \alpha^5 & \alpha & \alpha^3 \end{vmatrix}$ is equal to **[Kerala PET 2008]**

(A) $3\alpha^3$
(B) $3(\alpha^3 + \alpha^6 + \alpha^9)$
(C) $3(\alpha + \alpha^2 + \alpha^3)$
(D) 3

372. Let A and B be two matrices of order n × n. Let A be non-singular and B be singular. Consider the following:

1. AB is singular.
2. AB is non-singular.
3. $A^{-1}$ B is singular.
4. $A^{-1}$ B is non-singular.

Which of the above is/are correct?

(A) 1 only (B) 3 only
(C) 1 and 3 (D) 2 and 4

373. Let (*x*, *y*, z) be points with integer coordinates satisfying the system of homogeneous equations:

$3x - y - z = 0$ ....(i)
$-3x + z = 0$ ....(ii)
$-3x + 2y + z = 0$ ....(iii)

Then, the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is **[IIT 2009]**

(A) 6 (B) 7
(C) 49 (D) none of these

374. If a = 1 + 2 + 4 + ... upto n terms,
b = 1 + 3 + 9 + ... upto n terms
and c = 1 + 5 + 25 + ... upto n terms,

then $\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix}$ =

(A) $(30)^n$ (B) $(10)^n$
(C) 0 (D) $2^n + 3^n + 5^n$

375. Consider the system of equations in *x*, *y*, z as

$x \sin 3\theta - y + z = 0$
$x \cos 2\theta + 4y + 3z = 0$
$2x + 7y + 7z = 0$

If this system has a non-trivial solution, then for any integer n, values of θ are given by **[DCE 2009]**

(A) $\left(n + \frac{(-1)^n}{3}\right)\pi$ (B) $\left(n + \frac{(-1)^n}{4}\right)\pi$

(C) $\left(n + \frac{(-1)^n}{6}\right)\pi$ (D) $\frac{n\pi}{2}$

376. If A is an 3 × 3 non-singular matrix such that AA′ = A′A and B = $A^{-1}$A′, then BB′ equals **[JEE (Main) 2014]**

(A) $B^{-1}$ (B) $(B^{-1})'$
(C) I + B (D) I

377. If α, β ≠ 0 and f(n) = $\alpha^n + \beta^n$ and

$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= K(1-\alpha)^2 (1-\beta)^2 (\alpha-\beta)^2$, then K is equal to **[JEE (Main) 2014]**

(A) 1 (B) –1
(C) αβ (D) $\frac{1}{\alpha\beta}$

## Answers to Multiple Choice Questions

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. (D) | 2. (A) | 3. (A) | 4. (A) | 5. (C) | 6. (B) | 7. (D) | 8. (B) | 9. (B) | 10. (C) |
| 11. (B) | 12. (B) | 13. (C) | 14. (A) | 15. (B) | 16. (B) | 17. (D) | 18. (D) | 19. (D) | 20. (D) |
| 21. (D) | 22. (A) | 23. (A) | 24. (C) | 25. (A) | 26. (B) | 27. (D) | 28. (C) | 29. (A) | 30. (A) |
| 31. (C) | 32. (D) | 33. (B) | 34. (A) | 35. (B) | 36. (D) | 37. (A) | 38. (A) | 39. (C) | 40. (B) |
| 41. (B) | 42. (D) | 43. (A) | 44. (B) | 45. (A) | 46. (B) | 47. (C) | 48. (C) | 49. (D) | 50. (A) |
| 51. (A) | 52. (C) | 53. (B) | 54. (B) | 55. (D) | 56. (B) | 57. (C) | 58. (A) | 59. (B) | 60. (D) |
| 61. (B) | 62. (A) | 63. (B) | 64. (B) | 65. (D) | 66. (A) | 67. (B) | 68. (A) | 69. (C) | 70. (B) |
| 71. (A) | 72. (A) | 73. (C) | 74. (A) | 75. (B) | 76. (A) | 77. (B) | 78. (A) | 79. (B) | 80. (A) |
| 81. (D) | 82. (A) | 83. (B) | 84. (B) | 85. (D) | 86. (D) | 87. (B) | 88. (C) | 89. (B) | 90. (C) |
| 91. (C) | 92. (D) | 93. (D) | 94. (A) | 95. (C) | 96. (B) | 97. (A) | 98. (A) | 99. (D) | 100. (B) |
| 101. (A) | 102. (C) | 103. (D) | 104. (D) | 105. (A) | 106. (B) | 107. (A) | 108. (C) | 109. (C) | 110. (D) |
| 111. (D) | 112. (C) | 113. (D) | 114. (D) | 115. (B) | 116. (A) | 117. (D) | 118. (D) | 119. (B) | 120. (D) |
| 121. (D) | 122. (A) | 123. (A) | 124. (A) | 125. (D) | 126. (B) | 127. (A) | 128. (B) | 129. (D) | 130. (A) |
| 131. (C) | 132. (B) | 133. (A) | 134. (C) | 135. (C) | 136. (B) | 137. (B) | 138. (C) | 139. (B) | 140. (D) |
| 141. (B) | 142. (A) | 143. (C) | 144. (A) | 145. (D) | 146. (D) | 147. (D) | 148. (C) | 149. (D) | 150. (B) |
| 151. (B) | 152. (D) | 153. (C) | 154. (A) | 155. (C) | 156. (D) | 157. (C) | 158. (A) | 159. (C) | 160. (C) |
| 161. (C) | 162. (C) | 163. (B) | 164. (B) | 165. (A) | 166. (B) | 167. (A) | 168. (C) | 169. (B) | 170. (D) |
| 171. (B) | 172. (C) | 173. (A) | 174. (B) | 175. (A) | 176. (B) | 177. (A) | 178. (A) | 179. (D) | 180. (D) |
| 181. (D) | 182. (B) | 183. (B) | 184. (B) | 185. (C) | 186. (A) | 187. (A) | 188. (A) | 189. (B) | 190. (B) |
| 191. (D) | 192. (B) | 193. (D) | 194. (B) | 195. (C) | 196. (D) | 197. (D) | 198. (B) | 199. (A) | 200. (C) |
| 201. (B) | 202. (B) | 203. (D) | 204. (B) | 205. (B) | 206. (C) | 207. (C) | 208. (D) | 209. (D) | 210. (B) |
| 211. (D) | 212. (B) | 213. (D) | 214. (D) | 215. (D) | 216. (A) | 217. (C) | 218. (B) | 219. (B) | 220. (B) |
| 221. (B) | 222. (B) | 223. (C) | 224. (A) | 225. (B) | 226. (D) | 227. (A) | 228. (C) | 229. (D) | 230. (B) |
| 231. (A) | 232. (A) | 233. (A) | 234. (C) | 235. (D) | 236. (A) | 237. (B) | 238. (D) | 239. (B) | 240. (B) |
| 241. (C) | 242. (C) | 243. (C) | 244. (B) | 245. (B) | 246. (C) | 247. (C) | 248. (A) | 249. (A) | 250. (C) |
| 251. (A) | 252. (B) | 253. (B) | 254. (B) | 255. (A) | 256. (D) | 257. (A) | 258. (B) | 259. (A) | 260. (A) |
| 261. (B) | 262. (B) | 263. (C) | 264. (D) | 265. (A) | 266. (B) | 267. (B) | 268. (D) | 269. (D) | 270. (A) |
| 271. (A) | 272. (D) | 273. (B) | 274. (A) | 275. (A) | 276. (C) | 277. (A) | 278. (B) | 279. (C) | 280. (D) |
| 281. (B) | 282. (A) | 283. (D) | 284. (D) | 285. (B) | 286. (D) | 287. (A) | 288. (A) | 289. (A) | 290. (B) |
| 291. (D) | 292. (B) | 293. (D) | 294. (B) | 295. (C) | 296. (D) | 297. (A) | 298. (A) | 299. (D) | 300. (C) |
| 301. (A) | 302. (A) | 303. (C) | 304. (A) | 305. (B) | 306. (B) | 307. (A) | 308. (A) | 309. (B) | 310. (B) |
| 311. (A) | 312. (A) | 313. (B) | 314. (A) | 315. (D) | 316. (A) | 317. (B) | 318. (B) | 319. (B) | 320. (D) |
| 321. (A) | 322. (A) | 323. (A) | 324. (C) | 325. (C) | 326. (C) | 327. (D) | 328. (B) | 329. (A) | 330. (A) |
| 331. (D) | 332. (A) | 333. (B) | 334. (A) | 335. (A) | 336. (B) | 337. (B) | 338. (D) | 339. (A) | 340. (A) |
| 341. (B) | 342. (B) | 343. (B) | 344. (D) | 345. (B) | 346. (A) | 347. (B) | 348. (A) | 349. (C) | 350. (B) |
| 351. (A) | 352. (D) | 353. (B) | 354. (A) | 355. (C) | 356. (B) | 357. (A) | 358. (A) | 359. (A) | 360. (D) |
| 361. (B) | 362. (B) | 363. (C) | 364. (A) | 365. (D) | 366. (D) | 367. (A) | 368. (B) | 369. (C) | 370. (D) |
| 371. (D) | 372. (C) | 373. (B) | 374. (C) | 375. (C) | 376. (D) | 377. (A) | | | |

$$= 2^2 \times 2^3 \times 2^4 \begin{vmatrix} a_{11} & 2a_{12} & 2^2 a_{13} \\ a_{21} & 2a_{22} & 2^2 a_{23} \\ a_{31} & 2a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$= 2^9 \times 2 \times 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} P$$

$\therefore$ $|Q| = 2^{12}|P| = 2^{12} \times 2 = 2^{13}$

361. $\begin{vmatrix} a & b & a\alpha - b \\ b & c & b\alpha - c \\ 2 & 1 & 0 \end{vmatrix} = 0$

$\Rightarrow a[-(b\alpha - c)] - b[-2(b\alpha - c)] + (a\alpha - b)(b - 2c) = 0$

$\Rightarrow -ab\alpha + ac + 2b^2\alpha - 2bc + ab\alpha - 2ac\alpha - b^2 + 2bc = 0$

$\Rightarrow ac + 2b^2\alpha - 2ac\alpha - b^2 = 0$

$\Rightarrow (ac - b^2) - 2\alpha(ac - b^2) = 0$

$\Rightarrow (ac - b^2)(1 - 2\alpha) = 0$

$\Rightarrow ac - b^2 = 0$ or $1 - 2\alpha = 0$

Since, $\alpha \neq \frac{1}{2}$

$\therefore$ $b^2 = ac$ i.e., a, b, c are in G.P.

362. By Cramer's rule,

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} -d_1 & b_1 & c_1 \\ -d_2 & b_2 & c_2 \\ -d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\Rightarrow x = \frac{-\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\Rightarrow x = \frac{-\Delta(bcd)}{\Delta(abc)}$$

363. Let $M = \begin{bmatrix} a & b & c \\ x & y & z \\ l & m & n \end{bmatrix}$. Then,

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ y \\ m \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ by the equality of matrices,
$b = -1, y = 2, m = 3$

$$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a-b \\ x-y \\ l-m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ by the equality of matrices,
$a - b = 1, x - y = 1, l - m = -1$
$\Rightarrow a = 0, x = 3, l = 2$

$$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} a+b+c \\ x+y+z \\ l+m+n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$\therefore$ by the equality of matrices,
$a + b + c = 0, x + y + z = 0, l + m + n = 12$
$\Rightarrow c = 1, z = -5, n = 7$

$\therefore$ sum of diagonal elements of M
$= a + y + n$
$= 0 + 2 + 7 = 9$

364. $P.P^T = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $P.P^T = I$
$\Rightarrow P^T = P^{-1}$
Given, $Q = PAP^T$
$\Rightarrow P^TQP = P^T(PAP^T)P$
$= (P^TP)A(P^TP)$
$= A$ ....[$\because P^TP = I$]
$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $Q^2 = (PAP^T)(PAP^T)$
$= PA(P^TP)AP^T$
$= PA^2P^T$

$\therefore$ $P^TQ^2P = P^T(PA^2P^T)P$
$= (P^TP)A^2(P^TP)$
$= A^2$
$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Proceeding in this manner, we get

$P^TQ^{2005}P = A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

365. $P^T = 2P + I$

$\Rightarrow (P^T)^T = (2P + I)^T$

$\Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I$

$\Rightarrow P = 4P + 3I$

$\Rightarrow 3P + 3I = O$

$\Rightarrow P + I = O$

$\Rightarrow P = -I$

$\Rightarrow PX = -IX$

$\Rightarrow PX = -X$

366. $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow AB = BA$ ….(i)

Now, $(AB)^T = (-1)^n AB$

$\Rightarrow B^T A^T = (-1)^n AB$

$\Rightarrow (-B)A = (-1)^n AB$

....[$\because B^T = -B$ and $A^T = A$]

$\Rightarrow -BA = (-1)^n BA$ ….[From (i)]

$\Rightarrow (-1)^n = -1$

$\Rightarrow$ n is an odd natural number

367. Since, A is an upper triangular matrix.

$\therefore$ $|A| = 5 \times \alpha \times 5 = 25\alpha$

Given, $|A^2| = 25$

$\Rightarrow |A|^2 = 25 \Rightarrow (25\alpha)^2 = 25$

$\Rightarrow \alpha^2 = \dfrac{1}{25}$

$\Rightarrow |\alpha| = \dfrac{1}{5}$

368. $f(\alpha) = \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & \alpha^2 & 1 \\ \alpha^2 & 1 & \alpha \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$f(\alpha) = \begin{vmatrix} 1+\alpha+\alpha^2 & \alpha & \alpha^2 \\ 1+\alpha+\alpha^2 & \alpha^2 & 1 \\ 1+\alpha+\alpha^2 & 1 & \alpha \end{vmatrix}$

$= (1 + \alpha + \alpha^2) \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & 1 \\ 1 & 1 & \alpha \end{vmatrix}$

$= (1 + \alpha + \alpha^2)\{1(\alpha^3 - 1) - \alpha(\alpha - 1) + \alpha^2(1 - \alpha^2)\}$

$= (1 + \alpha + \alpha^2)(\alpha^3 - 1 - \alpha^2 + \alpha + \alpha^2 - \alpha^4)$

$= (1 + \alpha + \alpha^2)(\alpha - 1 + \alpha^3 - \alpha^4)$

$= (1 + \alpha + \alpha^2)\{(\alpha - 1) + \alpha^3(1 - \alpha)\}$

$= (1 + \alpha + \alpha^2) (\alpha - 1) (1 - \alpha^3)$

$= (\alpha^3 - 1) (1 - \alpha^3)$

$\therefore$ $f(3^{1/3}) = (3 - 1) (1 - 3)$

$= -4$ ....[$\because \alpha = 3^{1/3}$, $\therefore$ $\alpha^3 = 3$]

369. $|M_r| = \begin{vmatrix} r & r-1 \\ r-1 & r \end{vmatrix}$

$= r^2 - (r - 1)^2$

$= 2r - 1$

$\therefore$ $\det(M_1) + \det(M_2) + .... + \det(M_{2008})$

$= [2(1) - 1] + [2(2) - 1] + .... + [2(2008) - 1]$

$= 1 + 3 + 5 + ....$upto 2008 terms

$= (2008)^2$

370. Here, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

Only option (D) satisfies these relations.

371. $\begin{vmatrix} \alpha & \alpha^3 & \alpha^5 \\ \alpha^3 & \alpha^5 & \alpha \\ \alpha^5 & \alpha & \alpha^3 \end{vmatrix} = \alpha^3 \begin{vmatrix} 1 & \alpha^2 & \alpha^4 \\ \alpha^2 & \alpha^4 & 1 \\ \alpha^4 & 1 & \alpha^2 \end{vmatrix}$

$= \alpha^3\{1(\alpha^6 - 1) - \alpha^2(\alpha^4 - \alpha^4) + \alpha^4(\alpha^2 - \alpha^8)\}$

$= \alpha^3 (\alpha^6 - 1 + \alpha^6 - \alpha^{12})$

$= \alpha^3 (2\alpha^6 - 1 - \alpha^9.\alpha^3)$

$= 2\alpha^9 - \alpha^3 - \alpha^9\alpha^6$

$= 2 - \alpha^3 - \alpha^6$ ....[$\because \alpha^9 = 1$ (given)]

$= 2 - (-1)$ ....$\left[\begin{array}{l} \because \alpha^9 = 1, \therefore (\alpha^3)^3 - 1 = 0 \\ \Rightarrow (\alpha^3 - 1)(\alpha^6 + \alpha^3 + 1) = 0 \\ \Rightarrow \alpha^6 + \alpha^3 + 1 = 0 \text{ as } \alpha^3 \neq 1 \end{array}\right]$

$= 3$

372. Given, $|A| \neq 0$ and $|B| = 0$

$\therefore$ $|AB| = |A|\,|B| = 0$

and $|A^{-1} B| = |A^{-1}|\,|B|$

$= \dfrac{1}{|A|}|B|$ ....$\left[\because |A^{-1}| = \dfrac{1}{|A|}\right]$

$= 0$

$\therefore$ AB and $A^{-1}$ B are singular.

373. Adding (i) and (ii), we get $y = 0$.

From (ii), $z = 3x$.

Putting these values in $x^2 + y^2 + z^2 \leq 100$, we get $x^2 \leq 10$

$\Rightarrow -\sqrt{10} \leq x \leq \sqrt{10}$

Since, $x$ is an integer.

$\therefore$ $x = \pm3, \pm2, \pm1, 0$

Hence, there are 7 points.

374. Here, $a = \frac{1(2^n - 1)}{2-1} \Rightarrow a = 2^n - 1$

$b = \frac{1(3^n - 1)}{3-1} \Rightarrow 2b = 3^n - 1$

$c = \frac{1(5^n - 1)}{5-1} \Rightarrow 4c = 5^n - 1$

$\therefore \begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix} = \begin{vmatrix} 2^n-1 & 3^n-1 & 5^n-1 \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix}$

By taking 2 common from $R_2$, we get

$\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix} = 2\begin{vmatrix} 2^n-1 & 3^n-1 & 5^n-1 \\ 1 & 1 & 1 \\ 2^n & 3^n & 5^n \end{vmatrix}$

Applying $R_3 \to R_3 - R_1$, we get

$\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix} = 2\begin{vmatrix} 2^n-1 & 3^n-1 & 5^n-1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 0$ ....[$\because R_2 \equiv R_3$]

375. The given system of equations has a non-trivial solution.

$\therefore \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$

$\Rightarrow \sin 3\theta(7) + 1(7\cos 2\theta - 6) + 1(7\cos 2\theta - 8)$

$\Rightarrow 7 \sin 3\theta + 14\cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow 3 \sin\theta - 4 \sin^3\theta + 2 - 4 \sin^2 \theta - 2 = 0$

$\Rightarrow \sin \theta(4 \sin^2 \theta + 4 \sin \theta - 3) = 0$

$\Rightarrow \sin \theta (2 \sin \theta + 3) (2 \sin \theta - 1) = 0$

$\Rightarrow \sin \theta = 0$ or $\sin \theta = \frac{1}{2}$

$\Rightarrow \theta = n\pi$ or $\theta = n\pi + (-1)^n \frac{\pi}{6}$

376. $BB' = (A^{-1}A')(A^{-1}A')'$

$= (A^{-1}A')(A(A^{-1})')$

$= A^{-1} AA'(A^{-1})'$

....[$\because AA' = A'A$ (given)]

$= (A^{-1}A)(A'(A^{-1})')$

$= I(A^{-1} A)'$

$= I.I = I^2 = I$

377. $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$

$= \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$

$= (1 - \alpha)^2(1 - \beta)^2(\alpha - \beta)^2$

$\therefore$ K = 1